A Very Brief History of Calculus
Mathematics vs. the History of Mathematics

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- These slides do not do justice to the history of calculus, nor do they explain calculus to someone who does not already know it, but hopefully they highlight the fact that the history of calculus is interesting, and give some historical background for the material in an introductory real analysis course.
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Calculus as we know it today was the result of a long evolution, not a revolution, though Newton and Leibniz were certainly the central figures.
A Few Sources

- The MacTutor History of Mathematics Archive
  http://www-history.mcs.st-andrews.ac.uk/
- Euclid, *The Elements*
  http://aleph0.clarku.edu/~djoyce/java/elements/toc.html
- Google Books
  http://books.google.com/
Key Topics in a Calculus Course

★ Limits
★ Derivatives
★ Integrals

A Very Brief History of Calculus
Key Topics in a Real Analysis Course

- The real numbers
- Limits
- Derivatives
- Integrals
Historical Order

- Integrals
- Derivatives
- Limits
- The real numbers

A Very Brief History of Calculus
Related Topics with Long Histories

- Series (including power series)
- Algebra
- Analytic geometry
- Trigonometry
Ancient Greece

- Number (which are whole numbers) were distinct from magnitude (for example lengths of line segments)
- Irrational numbers were known, but understood only as magnitudes
- Infinite processes were avoided
- Tangent line questions were not widely studied
- The formula for the partial sums of a geometric series was known
- The ancient Greeks did not invent geometry, but contributed the idea of proof in geometry, and produced a large body of theorems and proofs
Area and Volume in Ancient Greece

- Area and volume formulas for various simple shapes were known throughout the ancient world.
- The ancient Greeks contributed the idea of proving results about area and volume, and in particular developed the method of exhaustion, attributed to Eudoxus of Cnidus (408–355 BCE).
- The ancient Greeks did not understand area and volume as we do today.
- Areas and volumes were not given by numerical values.
- The areas or volumes of different regions, or the ratios of such areas or volumes, were compared.
Euclid (c. 325–c. 265 BCE)

- Wrote “The Elements,” which is one of the most important mathematics texts ever written
- Gave a theory of ratios of magnitudes in Book V, and a separate theory of ratios of numbers in Book VII
- Had various area and volume results

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Euclid on Areas of Circles

- Book XII, Proposition 2: *Circles are to one another as the squares on their diameters*
- The proof is by the method of exhaustion
- Use proof by contradiction, looking at two cases
Euclid on Areas of Circles

Book X, Proposition 1: Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out.

Book XII, Proposition 1: Similar polygons inscribed in circles are to one another as the squares on their diameters.
Archimedes (287–212 BCE)

- Computed areas and volumes using the method of exhaustion
- Discussed tangent lines to what we call the Archimedean spiral
- Estimated $\pi$, and proved a geometric equivalent of the area formula for a circle
- Showed a geometric equivalent of the sum $\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}$

A Very Brief History of Calculus
Aristotle’s views were dominant

There was philosophical discussion of the infinite

Very little progress on tangent problems and area problems

A few infinite series were considered

Decimal notation for integers was imported from the Arab world
Nicole Oresme (1323–1382)

- Made an initial step toward analytic geometry around 1350
- Suggested the idea of a mathematical indivisible
- Understood that the area under the graph of the velocity of an object represents the distance travelled, which is the essential idea of the Fundamental Theorem of Calculus
- Proved that the harmonic series is divergent
Middle Ages—East

- Decimal notation was developed
- Irrational numbers were understood to be numbers
- Algebra was developed
- Trigonometry was developed
Renaissance

- Wider use of decimal notation for numbers
- Growth of algebra
- Move away from Aristotle
Calculus is one of the most widely useful, if not the most widely useful, mathematical tool for understanding the real world.
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And yet, the development of calculus required a willingness to use ideas, such as infinitesimals and limits, that are idealized and do not exist in the real world.
François Viète (1540–1603)

- Developed an approach to the study of equations that focused on general cases rather than specific examples.
- Promoted the use of symbols for variables and constants, which we take for granted today, but which was an innovation at the time.
Simon Stevin (1548–1620)

- Promoted the decimal place-value system for whole numbers
-Introduced the use of decimal fractions to the West, though he used only finite decimals
- Stated explicitly, perhaps for the first time, that there is no distinction between numbers and magnitudes
- Made a step toward the development of limits, to avoid the reductio ad absurdum argument of the method of exhaustion
17th Century

- Many developments in mathematics
- Growing interest in practical uses of mathematics
- Decline in adherence to ancient Greek methods
- Use of infinitesimals and indivisibles
- Use of series, including power series
Johannes Kepler (1571–1630)

- Computed volumes of solids of revolution, for the practical purpose of finding volumes of wine casks
- Focused on getting results rather than using Archimedean proofs
- Used infinitesimals freely to obtain his results
Bonaventura Cavalieri (1598–1647)

- Took the concept of indivisible and made it into a workable tool for finding areas and volumes
- Viewed planar regions as made up of infinitely many slices by parallel lines, and solid regions as made up of infinitely many slices by parallel planes, and compared the areas or volumes of two regions by comparing their slices
René Descartes (1596–1650)

- Invented analytic geometry
- Helped promote the recognition of all lengths of line segments as numbers
- Developed a method for finding tangent lines to some curves
Pierre de Fermat (1601–1665)

- Invented analytic geometry
- Found the maximum and minimum values of curves by considering what we write as \( \frac{f(x+e) - f(x)}{e} \), dividing as if \( e \) were non-zero, and then dropping \( e \) as if were zero
- Evaluated what we write as \( \int_0^a x^{\frac{p}{q}} \, dx \)
- Was among the first to notice, though only in special cases, a link between tangent problems and area problems

A Very Brief History of Calculus
Johann Hudde (1628–1704)

- Discovered algorithmic rules for computing the slopes of the tangent lines of arbitrary algebraic curves
Isaac Barrow (1630–1677)

- Computed slopes of tangent lines by implicitly using the idea of approximating tangent lines with secant lines, and dropping higher powers of infinitesimals
- Had geometric statements of both versions of the Fundamental Theorem of Calculus
- Did not exploit this understanding to provide a method for computing areas under curves
Read in the University of Cambridge, by Isaac Barrow, D.D. Professor of Mathematics, and Master of Trinity-College, etc.

Translated from the Latin Edition, revised, corrected and amended by the late Sir Isaac Newton.

By Edmund Stone, F.R.S.

LONDON, Printed for Stephen Austen, at the Angel and Bible in St. Paul's Church-Yard.

MDCCXXXV.

II. Let there be any Line ZGE, whose Axis is VD; and first, let the perpendicular Ordinates VZ, PG, DE to it, any how increase continually from the first VZ; also let the Line VIF be such, that drawing any right Line EDF perpendicular to VD (which cuts the Curves in the Points E, F, and VD in D,) the Rectangle under DF and any given right Line R, may be respectively equal to the intercepted Space VDEZ; and make DE: DF :: R: DT, and joyn the right Line TF: this will touch the Curve VIF.
\[ DF \times R = \text{area}(VDEZ) \]

\[ g(x) = \frac{1}{R} \int_a^x f(t) \, dt \]

\[ \frac{DE}{DF} = \frac{R}{DT} \]

\[ \frac{DF}{DT} = \frac{DE}{R} \]

\[ g'(x) = \frac{1}{R} f(x) \]
Calculus

- Recognizes derivatives and integrals as the main concepts
- Provides simple algorithmic methods
- Exploits the relationship between area problems and tangent problems
- Is widely applicable
- None of Newton’s and Leibniz’ predecessors had all that
Newton and Leibniz

- Worked geometrically with curves, rather than functions
- Used infinitesimals, rather than limits
- Took many known ingredients put them together into something new
- Had different approaches, and apparently worked independently
- Newton conceived of calculus first, and was the greater mathematician
- Leibniz’ published calculus first, and his approach and notation had more immediate impact
Isaac Newton (1643–1727)

- Worked out the basics of calculus in 1665-1666, with further discoveries later
- Based his approach to derivatives on the velocities of the $x$ and $y$ components of a point moving along a curve, denoted $\dot{x}$ and $\dot{y}$ and defined the derivative as the ratio of $\dot{y}$ and $\dot{x}$
- Calculated the derivative of any algebraic curve by using infinitesimals and implicit differentiation
Isaac Newton (1643–1727)

- Worked out the Chain Rule, and showed how to take derivatives of products and quotients
- Used an intuitive argument to show what we would phrase by saying that if $A$ is the area under the curve $y = f(x)$ then $\frac{dA}{dx} = y$, which is the Fundamental Theorem of Calculus
- Computed a table of antiderivatives, in part using integration by substitution, where some of the antiderivatives were given explicitly and others were handled with the binomial series (which he had discovered before calculus)
- Used these ideas to solve some area problems essentially as we do today, and that was the birth of calculus
Isaac Newton (1643–1727)

- Found maxima and minima by setting the derivative equal to zero and solving
- Worked out the arc length formula and computed some examples
- Used indefinite integrals, though he solved area problems with them
- Found the power series for $e^x$, $\sin x$ and $\cos x$
- Gave the first explicit (though not published) statement of the general formula for Taylor series, though without proof
- Used infinitesimals initially, but subsequently used something approaching the idea of a limit
Gottfried von Leibniz (1646–1716)

- Worked out his version of calculus in 1675–1677
- Based his approach to derivatives on infinitesimal changes in $x$ and $y$, which he denoted $dx$ and $dy$, and defined the derivative as the ratio of $dy$ and $dx$
- Appeared to be ambivalent about whether or not to think of $dx$ and $dy$ as infinitesimals
- Did not develop an approach resembling limits as did Newton
Gottfried von Leibniz (1646–1716)

- Worked out the derivative of power functions, as well as the Product Rule and Quotient Rule, writing everything in terms of differentials rather than derivatives.
- Observed that $dv$ is positive when $v$ is increasing, and analogously for decreasing, and hence that local extrema occur only when $dv = 0$.
- Gave a derivation of Snell’s Law of Refraction (which was already known at the time).
- Viewed integrals as a type of sum, and differentials as a type of difference, and hence integrals and differentials had an inverse relation.
- Stated the Fundamental Theorem of Calculus.
Gottfried von Leibniz (1646–1716)

- Had the formula for arc length
- Gave power series for $\ln(1+x)$, $\arctan x$, $\sin x$, $\cos x - 1$ and $e^x - 1$
- Used power series to solve differential equations
18th Century

- New developments in calculus and its applications
- Lack of rigor
- Power series were treated formally
- Divergent series were used
- Critique of calculus and responses to the critique
- The function concept was developed, though at first only continuous functions were considered
- Lack of clarity of the integral as definite or indefinite
Leonhard Euler (1707–1783)

- Used infinitely large and infinitely small numbers freely
- Clarified the notion of function
- Gave the first systematic treatment of logarithms and the trigonometric functions as we know them today
- Proved many results about series and power series
- Made advances in the study of differential equations
Jean d’Alembert (1717–1783)

- Proposed that the derivative be viewed as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, rather than as Newton’s ratio of fluxions or Leibniz’s ratio of differentials, though he did not have a rigorous definition of limits.
Joseph-Louis Lagrange (1736–1813)

- Attempted to avoid both infinitesimals and limits by viewing all functions as power series, and then picking off the derivative as a certain coefficient in such series.
- It was subsequently shown by Cauchy that not every differentiable function can be written as a power series.
- Introduced the term “derivative” and the notation $f'(x)$. 

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Joseph Fourier (1768–1830)

- Recognized that integrals of discontinuous functions were needed
- Changed the focus from indefinite integrals to definite integrals, and introduced the notation $\int_{a}^{b} f(x) \, dx$
- Brought attention to the importance of the convergence of series via his work on trigonometric series

A Very Brief History of Calculus
19th Century

- The start of modern rigor
- The formulation of calculus as we know it today
Carl Friedrich Gauss (1777–1855)

- Used the idea of least upper bounds informally
- Said that caution was need when using infinite quantities, and that they should be used only if their use can be viewed in terms of limits
- Published the first paper that said that power series should be used only where they are convergent
Bernard Bolzano (1781–1848)

- Defined the real numbers using sequences of rational numbers
- Made use of the Least Upper Bound Property
- Proved the Intermediate Value Theorem implicitly using the Monotone Convergence Theorem
- Gave the first modern formulation of continuity, without $\varepsilon-\delta$
- Gave the first example of a continuous nowhere differentiable function
Augustin Louis Cauchy (1789–1857)

- Brought calculus into the form we use it today
- Defined derivatives in terms of limits
- Defined definite integrals (for continuous functions) separately from derivatives
- Used only convergent series, proved convergence theorems
- Promoted rigor, though he had some errors because he lacked a proper definition of the real numbers
Recognized the need to give a more precise definition of integrability to accommodate discontinuous functions

Gave our modern definition of integrals via Riemann sums, though without $\varepsilon$-$\delta$

Gave criteria that are equivalent to integrability

Had some gaps in rigor because he lacked a proper definition of the real numbers
Gave what was probably the first rigorous construction of the real numbers from the rational numbers, using Dedekind cuts

Gave the first axiomatic characterization of the natural numbers

Provided what was probably the first rigorous proof of the Monotone Convergence Theorem
Karl Weierstrass (1815–1897)

- Gave an early construction of the real numbers from the rational numbers
- Wanted to remove all geometric reasoning from real analysis
- Changed the view of limits from the previous notion of a “variable approaching” something
- Gave the modern definition of continuity, though without $\delta$
- Ended the use of infinitesimals in real analysis.
Second Irony of the Development of Calculus

- The ancient Greeks did not think in terms of variability
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- The development of calculus was based upon thinking in terms of variability
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- The ancient Greeks did not think in terms of variability.
- The development of calculus was based upon thinking in terms of variability.
- And yet, to make calculus rigorous, the idea of variability had to be abandoned once again.
Georg Cantor (1845–1918)

- Invented set theory
- Had the idea of constructing the real numbers from the rational numbers using Cauchy sequences
Eduard Heine (1821–1881)

- Provided a rigorous treatment of Cantor’s approach to constructing the real numbers from the rational numbers.
- Was the first person to use the $\varepsilon$-$\delta$ definition of continuity as we do now, based upon the ideas of Weierstrass.
- Distinguished continuous and uniformly continuous functions, and showed that a continuous function on a closed bounded interval is uniformly continuous.
Giuseppe Peano (1858–1932)

- Gave an axiomatic definition of the natural numbers, now called the Peano Postulates
David Hilbert (1862–1943)

- Gave an axiomatic definition of the real numbers
20th Century

- The basics of calculus were already known
- Generalizations of the Riemann integral
- Calculus in more general spaces
Henri Lebesgue (1875–1941)

- Proved a theorem that characterizes which functions are Riemann integrable
- Defined a new type of integral that agrees with the Riemann integral for functions that are Riemann integrable, which works for some functions that are not Riemann integrable, and which has some nicer properties than the Riemann integral
Abraham Robinson (1918–1974)

- Developed non-standard analysis, which used a system of numbers that contains the real numbers as well as infinitesimals and infinite numbers, and which is an alternative to the way real analysis is usually done via limits
The End