1. Let $\mathcal{R}$ be the tetrahedron in $\mathbb{R}^3$ with vertices $(0, 0, 1)$, $(0, 0, -1)$, $(1, 1, 0)$, and $(1, -1, 0)$. Evaluate the following integral:

$$\int \int \int_{\mathcal{R}} xy^2 \, dV$$

**Answer:** First, it helps to determine the 4 planes that determine the sides of the tetrahedron. The top plane, containing the points $(0, 0, 1), (1, 1, 0),$ and $(1, -1, 0)$, has equation:

$$x + z = 1$$

The bottom plane, containing the points $(0, 0, -1), (1, 1, 0),$ and $(1, -1, 0)$, has equation:

$$x - z = 1$$

The side plane that contains the points $(0, 0, 1), (0, 0, -1),$ and $(1, 1, 0)$ has equation:

$$x - y = 0$$

And the other side plane, containing the points $(0, 0, 1), (0, 0, -1),$ and $(1, -1, 0)$, has equation:

$$x + y = 0$$

Note that each of the above equations only depend on two variables. This makes it relatively easy to set up the integral. For example, if we integrate with respect to $z$ first, then the bounds for $z$ go from the bottom plane to the top place: $z = x - 1$ to $z = 1 - x$.

Then, we just need to consider the triangle in the $xy$-plane. Two sides of this triangle are given by the lines $x - y = 0$ and $x + y = 0$ (these come from the equations for the side planes of the tetrahedron). The third side of the triangle is the line $x = 1$. Thus:

$$\int \int \int_{\mathcal{R}} xy^2 \, dV = \int_0^1 \int_{-x}^{1-x} \int_{-1}^{1-x} xy^2 \, dz \, dy \, dx = \frac{2}{45}$$
2. Let $\mathcal{R}$ be the region in $\mathbb{R}^3$ inside the sphere of radius 2 centered at the origin and outside the cylinder of radius 1 centered about the $z$-axis.

(a) Use cylindrical coordinates to evaluate $\iiint_{\mathcal{R}} (x^2 + y^2) \, dV$.

(b) Use spherical coordinates to evaluate $\iiint_{\mathcal{R}} \frac{1}{x^2 + y^2 + z^2} \, dV$.

Answer:

(a) As the region is symmetrical about the $z$-axis, we know that the bounds for $\theta$ are from 0 to $2\pi$. To find the bounds for $r$ and $z$, it will help to look at the object in the $rz$-plane.

We could integrate with respect to either $z$ or $r$ first. If we use $z$ first, we get:

$$
\iiint_{\mathcal{R}} (x^2 + y^2) \, dV = \int_{0}^{2\pi} \int_{1}^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3 \, dz \, dr \, d\theta
$$

$$
= \int_{0}^{2\pi} \int_{1}^{2} \left[ z r^2 \right]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \, dr \, d\theta
$$

$$
= \int_{0}^{2\pi} \int_{1}^{2} 2 r^3 \sqrt{4-r^2} \, dr \, d\theta
$$

This integral has become somewhat hard to evaluate by hand. You could evaluate it using a computer, or you can instead integrate
with respect to $r$ first:

\[
\int \int \int_R (x^2 + y^2) \, dV = \int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \int_{1}^{\sqrt{3} - z^2} r^3 \, dr \, dz \, d\theta
\]

\[
= \int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \left[ \frac{1}{4} r^4 \right]_1^{\sqrt{3} - z^2} \, dz \, d\theta
\]

\[
= \int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \left( \frac{1}{4} (4 - z^2)^2 - \frac{1}{4} \right) \, dz \, d\theta
\]

\[
= \int_0^{2\pi} \frac{22\sqrt{3}}{5} \, d\theta = \frac{44\pi \sqrt{3}}{5}
\]

This is the same answer that we obtain if we use a computer to evaluate the first previous integral.

(b) As in part (a), it helps to look at the object in the $rz$-plane (see the picture from part (a)). If we integrate with respect to $\rho$ first, the bounds for $\rho$ go from $r = 1$ to $\rho = 2$. Since $r = \rho \sin \phi$, the lower bound for $\rho$ is $\rho = \frac{1}{\sin \phi} = \csc \phi$. Then, the bounds for $\phi$ are from $\phi = \frac{\pi}{6}$ to $\phi = \frac{5\pi}{6}$. Thus:

\[
\int \int \int_R \frac{1}{x^2 + y^2 + z^2} \, dV = \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\csc \phi}^{2} \left( \frac{1}{\rho^2} \right) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\csc \phi}^{2} \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \left[ \rho \sin \phi \right]_{\csc \phi}^{2} \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} (2 \sin \phi - 1) \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \left( 2\sqrt{3} - \frac{2\pi}{3} \right) \, d\theta = \frac{4\pi \sqrt{3} - 4\pi^2}{3}
\]
3. Consider a cone of height 1 with base the unit circle on the $xy$-plane and with mass density $\delta(x, y, z) = z^2 \sqrt{x^2 + y^2}$. Find the center of mass of the cone.

**Answer:** The cone is symmetric about the $z$-axis. As in problem 2, it will be helpful to draw a cross-section in the $rz$-plane:

```
\begin{align*}
\text{Cylindrical coordinates will be the easiest to use for this problem. Since the object is symmetrical about the } z\text{-axis the bounds for } \theta \text{ are from } \theta = 0 \text{ to } \theta = 2\pi. \text{ It works to integrate with respect to either } z \text{ or } r \text{ first. If we integrate with respect to } r \text{ first, we obtain the following integral to compute the total mass of the object:}
\end{align*}
```

```
\text{Mass} = \int_0^{2\pi} \int_0^1 \int_0^{1-z} z^2 r^2 \, dr \, dz \, d\theta = \frac{\pi}{90}
```

To find the center of mass, note that both the object and the density function are symmetric with respect to $x$ and $y$, so $\bar{x} = 0$ and $\bar{y} = 0$. We just need to compute $\bar{z}$:

```
\bar{z} = \frac{1}{M} \int_0^{2\pi} \int_0^1 \int_0^{1-z} z^3 r^2 \, dr \, dz \, d\theta = \left( \frac{90}{\pi} \right) \left( \frac{\pi}{210} \right) = \frac{3}{7}
```

Thus, the center of mass of the object is $\left( 0, 0, \frac{3}{7} \right)$. 

4. Let \( \mathcal{R} \) be the region in \( \mathbb{R}^2 \) described by the parametric equations:

\[
x = u(3t^2 - u^2) \quad \text{and} \quad y = t(3u^2 - t^2)
\]

for \( 0 \leq t \leq 1 \) and \( 0 \leq u \leq 1 \).

(a) Draw a sketch of the region \( \mathcal{R} \), including the gridlines \( t = 0, t = 0.25, t = 0.5, t = 0.75, \) and \( t = 1 \), and \( u = 0, u = 0.25, u = 0.5, u = 0.75, \) and \( u = 1 \). (You can use a calculator or computer to graph the parametric curves.)

(b) Find the Jacobian \( J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix} \).

(c) Use the formula \( dA = |J| \, dt \, du \) to compute the area of the region \( \mathcal{R} \).

Answer:

(a) Here is the region:

\[
\begin{array}{c}
\begin{array}{c}
-1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0.5
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1.5
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
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\end{array}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
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\begin{array}{c}
\begin{array}{c}
0.5
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1
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2
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\end{array}
\begin{array}{c}
\begin{array}{c}
-1
\end{array}
\end{array}
\end{array}
\]

(b) \( J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 6tu & 3t^2 - 3u^2 \\ 3u^2 - 3t^2 & 6tu \end{vmatrix} = 9t^4 + 18t^2u^2 + 9u^4 \)

(c) Area = \( \int_0^1 \int_0^1 (9t^4 + 18t^2u^2 + 9u^4) \, dt \, du = \frac{28}{5} \)