Math 601 Solutions to Homework 7

1. Find a vector in the plane \( x_1 + 2x_2 - 3x_3 = 0 \) that is orthogonal to the vector \[
\begin{bmatrix}
-2 \\
3 \\
4
\end{bmatrix}
\].

**Answer:**
The dot product of the vector \((x_1, x_2, x_3)\) with \((-2, 3, 4)\) must be zero. This gives the equation:

\[-2x_1 + 3x_2 + 4x_3 = 0\]

We must therefore solve the following system of linear equations:

\[
\begin{align*}
x_1 + 2x_2 - 3x_3 &= 0 \\
-2x_1 + 3x_2 + 4x_3 &= 0
\end{align*}
\]

\[
x_1 = \frac{17}{7}x_3 \\
x_2 = \frac{2}{7}x_3
\]

\(x_3\) is free

Any vector satisfying these equations is a correct solution to this problem. The simplest solution is the vector \[\begin{bmatrix} 17 \\ 2 \\ 7 \end{bmatrix}\].

**Note:** The vector \((0, 0, 0)\) is also a correct solution to this problem as stated. The statement of the problem should have included the word “nonzero” to exclude this possibility.

2. Consider the following subspace of \(\mathbb{R}^5\):

\[
S = \text{Span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -3 \\ 2 \\ 3 \\ \end{bmatrix} \right\}
\]

(a) Find an orthonormal basis for \(S\).
(b) Find an orthonormal basis for \(S^\perp\).

**Answers:**
(a) We use the Gram-Schmidt process:

1. \(\|(2, 2, 1, 0, 0)\| = \sqrt{2^2 + 2^2 + 1^2} = 3\), so \(u_1 = \frac{1}{3}(2, 2, 1, 0, 0)\) is a unit vector.
2. \((5, 3, 2, 1, 1) \cdot \frac{1}{3}(2, 2, 1, 0, 0) = \frac{1}{3}(10 + 6 + 2) = 6\), so

\[(5, 3, 2, 1, 1) - (6)\frac{1}{3}(2, 2, 1, 0, 0) = (1, -1, 0, 1, 1)\] is orthogonal to \(u_1\).
3. \[ (2, -5, -3, 2, 3) \cdot \frac{1}{3} (2, 2, 1, 0, 0) = \frac{1}{3} (4 - 10 - 3) = -3, \] so 
\[ (2, -5, -3, 2, 3) \cdot \frac{1}{2} (1, -1, 0, 1, 1) = \frac{1}{2} (2 + 5 + 2 + 3) = 6, \] so 
\[ (2, -5, -3, 2, 3) - (-3) \frac{1}{3} (2, 2, 1, 0, 0) = (1, -1, 0, 1, 1) = (1, 0, -2, -1, 0) \]
is orthogonal to both \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \). 

\[ ||(1, 0, -2, -1, 0)|| = \sqrt{6}, \] so \( \mathbf{u}_3 = \frac{1}{\sqrt{6}} (1, 0, -2, -1, 0) \) is a unit vector orthogonal to both \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \).

Therefore, \[ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 0 \end{bmatrix} \] is an orthonormal basis for \( S \).

(b) A vector \((x_1, x_2, x_3, x_4, x_5)\) is an element of \( S^\perp \) if it is orthogonal to each of the three basis vectors for \( S \). This gives the following three equations:

\[
\begin{align*}
2x_1 + 2x_2 + x_3 &= 0 \\
5x_1 + 3x_2 + 2x_3 + x_4 + x_5 &= 0 \\
2x_1 - 5x_2 - 3x_3 + 2x_4 + 3x_5 &= 0
\end{align*}
\]

Therefore, \((-1, 2, -2, 3, 0)\) and \((-4, 5, -2, 0, 9)\) are a basis for \( S^\perp \). We can make these basis vectors orthonormal using the Gram-Schmidt process:

1. \[ ||(-1, 2, -2, 3, 0)|| = \sqrt{1 + 4 + 4 + 9} = \sqrt{18} = 3\sqrt{2}, \] so \( \mathbf{u}_1 = \frac{1}{3\sqrt{2}} (-1, 2, -2, 3, 0) \) is a unit vector.

2. \[ (-4, 5, -2, 0, 9) \cdot \frac{1}{3\sqrt{2}} (-1, 2, -2, 3, 0) = \frac{1}{3\sqrt{2}} (4 + 10 + 4) = 3\sqrt{2}, \] so 
\[ (-4, 5, -2, 0, 9) - (3\sqrt{2}) \frac{1}{3\sqrt{2}} (-1, 2, -2, 3, 0) = (-3, 3, 0, -3, 9) \] is orthogonal to \( \mathbf{u}_1 \).

\[ ||(-3, 3, 0, -3, 9)|| = 3||(-1, 1, 0, -1, 3)|| = 3\sqrt{12} = 6\sqrt{3}, \] so \( \mathbf{u}_2 = \frac{1}{6\sqrt{3}} (-3, 3, 0, -3, 9) = \frac{1}{2\sqrt{3}} (-1, 1, 0, -1, 3) \) is a unit vector orthogonal to \( \mathbf{u}_1 \).

Therefore, \[ \begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix} \] is an orthonormal basis for \( S^\perp \).
3. Let \( f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(nx) \), and let \( g(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} \cos(nx) \). Evaluate the following integrals:

(a) \( \int_{-\pi}^{\pi} f(x) \cos(5x) \, dx \)

(b) \( \int_{-\pi}^{\pi} f(x) g(x) \, dx \)

(c) \( \int_{-\pi}^{\pi} g(x) \sin^2 x \, dx \)

**Answers:**

Recall that the functions \( \frac{1}{\sqrt{2}} \), \( \sin nx \), and \( \cos nx \) (for \( n = 1, 2, 3, \ldots \)) are orthonormal with respect to the inner product:

\[
\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx
\]

We can express \( f(x) \) and \( g(x) \) in terms of this basis as follows:

\[
f(x) = (\sqrt{2}) \frac{1}{\sqrt{2}} + \frac{1}{2} \cos x + \frac{1}{2^2} \cos 2x + \frac{1}{2^3} \cos 3x + \frac{1}{2^4} \cos 4x + \cdots
\]

\[
g(x) = (\sqrt{2}) \frac{1}{\sqrt{2}} + \frac{1}{2} \cos x + \frac{1}{2^2} \cos 2x + \frac{1}{2^3} \cos 3x + \frac{1}{2^4} \cos 4x + \cdots
\]

(a) The inner product \( \langle f(x), \cos 5x \rangle \) is just the coefficient of \( \cos 5x \) in the expansion of \( f(x) \):

\[
\int_{-\pi}^{\pi} f(x) \cos(5x) \, dx = \pi \langle f(x), \cos 5x \rangle = \pi \left( \frac{1}{2^5} \right) = \pi \left( \frac{1}{32} \right)
\]

(b) We have:

\[
\int_{-\pi}^{\pi} f(x) g(x) \, dx = \pi \langle f(x), g(x) \rangle
\]

\[
= \pi \left( (\sqrt{2}) (\sqrt{2}) + \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) + \left( \frac{1}{2^2} \right) \left( \frac{1}{3^2} \right) + \cdots \right)
\]

\[
= \pi \left( 2 + \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \cdots \right)
\]

By the given formula:
Therefore, the integral is equal to $\frac{11\pi}{5}$.

(c) We can express $\sin^2 x$ in terms of our orthonormal basis as follows:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \quad \text{(half-angle identity)}$$

Therefore:

$$\int_{-\pi}^{\pi} g(x) \sin^2 x \, dx = \pi \langle g(x), \sin^2 x \rangle$$

$$= \pi \left( \sqrt{2} \left( \frac{1}{\sqrt{2}} \right) + \left( -\frac{1}{2} \right) \right)$$

$$= \frac{17\pi}{18}$$

4. On the interval $[-\pi, \pi]$, it is possible to express the function $f(x) = x$ as a Fourier series of the form:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

(a) Find a formula for the $n$th coefficient of $a_n$.

(b) Use your answer to part (a) to find the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (Hint: What is $\|f\|$?)
Answers:

(a) Using integration by parts (or a computer algebra system), we have:

\[
\langle f(x), \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx
\]

\[
= \frac{1}{\pi} \left[ \frac{\sin(nx)}{n^2} - \frac{x \cos(nx)}{n} \right]_{-\pi}^{\pi}
\]

But \( \sin(n\pi) = 0 \) for any integer \( n \), and \( \cos(n\pi) = \cos(-n\pi) = (-1)^n \). Therefore:

\[
\langle f(x), \sin(nx) \rangle = \frac{1}{\pi} \left[ \left( 0 - \frac{\pi(-1)^n}{n} \right) - \left( 0 - \frac{(-\pi)(-1)^n}{n} \right) \right]
\]

\[
= \frac{1}{\pi} \left( -\frac{2\pi(-1)^n}{n} \right)
\]

\[
= \frac{(-1)^{n+1} 2}{n}
\]

(b) Consider the quantity \( \|f\|^2 = \langle f, f \rangle \). There are two ways of computing this quantity:

**Calculation 1**  We can use the definition of the inner product:

\[
\langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2}{3} \pi^2
\]

**Calculation 2**  We can use the series the we computed in part (a):

\[
f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx)
\]

Using this series, we can compute \( \langle f, f \rangle \) by taking the sum of the squares of the coefficients:

\[
\langle f, f \rangle = \sum_{n=1}^{\infty} \left( (-1)^{n+1} \frac{2}{n} \right)^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}
\]
These two formulas for $\langle f, f \rangle$ must be equal:

$$
\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2}{3} \pi^2
$$

Dividing through by 4 gives

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
$$

This is a famous formula in mathematics. Finding the exact value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known as the **Basel problem**, and it was first proposed by the Italian mathematician Pietro Mengoli in 1644. The problem was not solved until nearly 100 years later by Leonhard Euler. (He solved it in 1735, using a different method than the one here.)