1. Use Cramer’s Rule to solve for $x_1$, $x_2$, and $x_3$ in terms of $a$ and $b$.

\[
ax_1 + bx_2 - x_3 = 0 \\
5x_1 + bx_2 - x_3 = 2 \\
3x_1 - x_2 + x_3 = 3
\]

Answer: From Cramer’s Rule, we get:

\[
x_1 = \frac{\begin{vmatrix} 0 & 1 & -1 \\ 2 & b & -1 \\ a & 1 & -1 \end{vmatrix}}{\begin{vmatrix} a & 1 & -1 \\ 5 & b & -1 \\ 3 & -1 & 1 \end{vmatrix}} = \frac{3b - 3}{ab - a + 3b - 3}
\]

\[
x_2 = \frac{\begin{vmatrix} a & 0 & -1 \\ 5 & 2 & -1 \\ a & 1 & -1 \end{vmatrix}}{\begin{vmatrix} a & 1 & -1 \\ 5 & b & -1 \\ 3 & -1 & 1 \end{vmatrix}} = \frac{5a - 9}{ab - a + 3b - 3}
\]

\[
x_3 = \frac{\begin{vmatrix} a & 1 & 0 \\ 5 & b & 2 \\ a & 1 & -1 \end{vmatrix}}{\begin{vmatrix} a & 1 & -1 \\ 5 & b & -1 \\ 3 & -1 & 1 \end{vmatrix}} = \frac{3ab + 2a - 9}{ab - a + 3b - 3}
\]

Thus:

\[
\begin{align*}
x_1 &= \frac{3b - 3}{ab - a + 3b - 3} \\
x_2 &= \frac{5a - 9}{ab - a + 3b - 3} \\
x_3 &= \frac{3ab + 2a - 9}{ab - a + 3b - 3}
\end{align*}
\]
2. Consider the subspace $S$ of $C[0,1]$ spanned by $\cosh x$ and $\sinh x$, and consider the following two bases of the subspace $S$:

$$
E = [\cosh x, \sinh x]\n$$

$$
F = [e^x, e^{-x}]\n$$

(a) Find the transition matrix from $E$ to $F$.
(b) Let $D$ be the differentiation operator on $S$; that is, $D(f(x)) = f'(x)$.

Find the matrix representing $D$ with respect to the basis $E$.

(c) Find the matrix representing $D$ with respect to the basis $F$.

Answer:

(a) We write each of the vectors in $E$ as a linear combination of the vectors in $F$:

$$
cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}\n$$

$$
\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}\n$$

Thus, the transition matrix from $E$ to $F$ is

$$
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
$$

Note: If you use the definition of change-of-basis matrix given in *Schaum’s Outline*, your answer to this question would be the inverse of the above matrix, which is:

$$
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
$$

(b) To find the matrix representing $D$ with respect to the basis $E$, we apply $D$ to each of the basis vectors:

$$
D(\cosh x) = \sinh x
$$

$$
D(\sinh x) = \cosh x
$$

Thus, the matrix representing $D$ with respect to the basis $E$ is

$$
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
$$
(c) To find the matrix representing $D$ with respect to the basis $F$, we apply $D$ to each of the basis vectors:

\[
D(e^x) = e^x \\
D(e^{-x}) = -e^{-x}
\]

Thus, the matrix representing $D$ with respect to the basis $F$ is

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

3. Consider the following linear transformation from $P_3$ to $P_2$:

\[L(p) = p'' + 2p'\]

Find the matrix representation of $L$ with respect to the bases \{1 + x, 1 - x, 1 + x^2\} and \{2 + x, x\}.

**Answer:** We apply $L$ to each of the vectors in the first basis:

\[
L(1 + x) = 2 \\
L(1 - x) = -2 \\
L(1 + x^2) = 2 + 4x
\]

Now, we write each of the above polynomials as a linear combination of the polynomials $2 + x$ and $x$. We get:

\[
2 = 1(2 + x) - 1(x) \\
-2 = -1(2 + x) + 1(x) \\
2 + 4x = 1(2 + x) + 3(x)
\]

Thus, the matrix representing $L$ with respect to the given bases is:

\[
\begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & 3
\end{bmatrix}
\]
4. Consider the following matrix:

\[ A = \begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix} \]

(a) Find a nonzero vector \( u \) such that \( Au = 2u \).
(b) Find a nonzero vector \( v \) such that \( Av = 3v \).
(c) Find the matrix representation for \( A \) with respect to the basis \( [u, v] \).

**Answer:**

(a) We want to find a vector \( u = \begin{pmatrix} a \\ b \end{pmatrix} \) such that

\[
\begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}
\]

Thus, we need \( a \) and \( b \) to satisfy the following equations:

\[
\begin{align*}
-3a + 10b &= 2a \\
-3a + 8b &= 2b
\end{align*}
\]

This is equivalent to:

\[
\begin{align*}
-5a + 10b &= 0 \\
-3a + 6b &= 0
\end{align*}
\]

Solving for \( a \) and \( b \), we get that \( a = 2b \). There are infinitely many solutions. We are just asked for one solution, so we can choose \( b = 1 \). Thus, one nonzero vector is

\[ u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

**Note:** Any multiple of this vector also works.

(b) We use the same steps as in part (a). We want to find a vector \( v = \begin{pmatrix} a \\ b \end{pmatrix} \) such that

\[
\begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}
\]

Thus, we need \( a \) and \( b \) to satisfy the following equations:

\[
\begin{align*}
-3a + 10b &= 3a \\
-3a + 8b &= 3b
\end{align*}
\]
This is equivalent to:

\[-6a + 10b = 0\]
\[-3a + 5b = 0\]

Solving for \(a\) and \(b\), we get that \(a = \frac{5}{3}b\). There are infinitely many solutions. We are just asked for one solution, so we can choose \(b = 3\). Thus, one nonzero vector is

\[
v = \begin{bmatrix} 5 \\ 3 \end{bmatrix}
\]

**Note:** Any multiple of this vector also works.

(c) We apply \(A\) to \(u\) and \(v\):

\[
\begin{bmatrix}
-3 & 10 \\
-3 & 8
\end{bmatrix}
\begin{bmatrix}
2 \\
1
\end{bmatrix} =
\begin{bmatrix}
4 \\
2
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 & 10 \\
-3 & 8
\end{bmatrix}
\begin{bmatrix}
5 \\
3
\end{bmatrix} =
\begin{bmatrix}
15 \\
9
\end{bmatrix}
\]

Now, we write these as linear combinations of \(u\) and \(v\). We get:

\[
\begin{bmatrix}
4 \\
2
\end{bmatrix} = 2u + 0v
\]

\[
\begin{bmatrix}
15 \\
9
\end{bmatrix} = 0u + 3v
\]

Thus, the matrix representing \(A\) with respect to the basis \([u, v]\) is

\[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\]

**Note 1:** This answer does not depend on which vectors we chose for parts (a) and (b).

**Note 2:** Since we knew that \(Au = 2u\) and \(Av = 3v\), we did not need to do any computations to obtain this matrix.