Math 601 Solutions to Homework 3

1. Consider a $4 \times 6$ matrix $A$ with rank 4.

(a) What is the dimension of the nullspace of $A$?
(b) Are the columns of $A$ linearly independent?
(c) Are the rows of $A$ linearly independent?
(d) Do the columns of $A$ span $\mathbb{R}^4$?
(e) Do the rows of $A$ span $\mathbb{R}^6$?
(f) Suppose that $b$ is a vector in $\mathbb{R}^4$. How many solutions are there to the equation $Ax = b$?

**Answer:** Since $A$ has rank 4 that means that the dimension of the column space is 4 (and also the dimension of the row space is 4). If we row reduce $A$, we’ll get a matrix with 4 pivots. There will be a pivot in each row, and four of the six columns will have pivots. The result will be something like:

$$A \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & a & b \\ 0 & 1 & 0 & 0 & c & d \\ 0 & 0 & 1 & 0 & e & f \\ 0 & 0 & 0 & 1 & g & h \end{pmatrix}$$

Or something like:

$$A \rightarrow \begin{pmatrix} 1 & a & 0 & b & 0 & 0 \\ 0 & 0 & 1 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Or something similar. This allows us to answer the questions.

(a) By the rank-nullity theorem, rank + nullity = 6, so the dimension of the nullspace is 2.
(b) There are 6 columns with dimension 4, so they are not linearly independent.
(c) There are 4 rows with dimension 4, so they are linearly independent.
(d) Yes, since the columns have dimension 4, they span $\mathbb{R}^4$.

(e) There are only 4 rows, so the rows do not span $\mathbb{R}^6$.

(f) A linear system always has either no solutions, 1 unique solution, or infinitely many solutions. Since there are free variables, there cannot be just 1 unique solution. If there were no solutions, the row reduced matrix would have a row of 0’s. Since, this is not the case, there are infinitely many solutions.

2. Consider the following linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$:

- $L_1$ rotates each vector $90^\circ$ about the $x$-axis (see the picture below).
- $L_2$ rotates each vector $45^\circ$ about the $y$-axis (see the picture below).
- $L_3$ first rotates $90^\circ$ about the $x$-axis and then rotates $45^\circ$ about the $y$-axis.
- $L_4$ first rotates $45^\circ$ about the $y$-axis, and then rotates $90^\circ$ about the $x$-axis.

(a) Find the matrix representations of $L_1, L_2, L_3,$ and $L_4$.

(b) Compute $L_3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$.

Answer:

(a) To find the matrix representations, we apply the linear transformations to each of the standard basis vectors, and the results are
the columns of the matrix. For $L_1$:

$$L_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad L_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad L_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Thus, the matrix representation of $L_1$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

For $L_2$:

$$L_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \quad L_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad L_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

Thus, the matrix representation of $L_2$ is $\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$.

We could do the same thing for $L_3$, but instead, we note that $L_3$ first applies $L_1$ and then applies $L_2$. Thus:

$$L_3 = L_2L_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

Similarly:

$$L_4 = L_1L_2 = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

(b) $L_3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} + \sqrt{2} \\ 3 \\ -1/\sqrt{2} + \sqrt{2} \end{pmatrix}$

3. Let $P_5$ denote the vector space of all polynomials with degree less than or equal to 4. Consider the subspace of $P_5$ consisting of polynomials $p(x)$ for which $p(1) = 0$ and $p(2) = 0$. Find a basis for this subspace.

**Answer:** A polynomial in $P_5$ is of the form $p(x) = a + bx + cx^2 + dx^3 + ex^4$. If the polynomial is in the subspace, then $p(1) = 0$ and $p(2) = 0$, which means:
\[
\begin{align*}
a + b + c + d + e &= 0 \\
a + 2b + 4c + 8d + 16e &= 0
\end{align*}
\]

If we solve these equations for \(a\) and \(b\), we get:
\[
\begin{align*}
a &= 2c + 6d + 14e \\
b &= -3c - 7d - 15e
\end{align*}
\]
Thus, every polynomial in the subspace can be written in the form:
\[
p(x) = (2c + 6d + 14e) + (-3c - 7d - 15e)x + cx^2 + dx^3 + ex^4
\]
We can rewrite this as:
\[
p(x) = c(2 - 3x + x^2) + d(6 - 7x + x^3) + e(14 - 15x + x^4)
\]
Thus, every polynomial in the subspace can be written as a linear combination of the polynomials \(2 - 3x + x^2, 6 - 7x + x^3,\) and \(14 - 15x + x^4\).

These polynomials are also linearly independent, so they form a basis. Thus, a basis for the subspace is \(\{2 - 3x + x^2, 6 - 7x + x^3, 14 - 15x + x^4\}\).

4. Let \(C[-\pi, \pi]\) denote the vector space of all real-valued functions that are defined and continuous on the closed interval \([-\pi, \pi]\).

(a) Consider the subspace of \(C[-\pi, \pi]\) spanned by the vectors \(\cos x, \sin x,\) and \(\sin(2x)\). What is the dimension of this subspace? Explain your answer.

(b) Consider the subspace of \(C[-\pi, \pi]\) spanned by the vectors \(\cos x, \cos(3x),\) and \(\cos^3 x\). What is the dimension of this subspace? Explain your answer.

\textbf{Answer:}

(a) The main thing we need to do is determine whether the vectors are linearly independent.
Suppose there exists \(c_1, c_2, c_3\) such that:
\[
c_1 \cos x + c_2 \sin x + c_3 \sin(2x) = 0
\]
This must hold for all values of \(x\), so we can plug in specific values of \(x\) and get linear equations involving \(c_1, c_2,\) and \(c_3\). We will plug in \(x = 0, x = \frac{\pi}{2},\) and \(x = \frac{\pi}{4}\):
\[
\begin{align*}
x &= 0 & c_1 &= 0 \\
x &= \frac{\pi}{2} & c_2 &= 0 \\
x &= \frac{\pi}{4} & \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}} + c_3 &= 0
\end{align*}
\]

Solving this system of linear equations, we get \( c_1 = 0, c_2 = 0, \) and \( c_3 = 0. \) Thus, the vectors \( \cos x, \sin x, \) and \( \sin(2x) \) are linearly independent. Since these vectors also span the subspace, they form a basis for the subspace. Thus, the dimension of the subspace is 3.

(b) Again, we need to determine if the vectors are linearly independent. In this case, the vectors are linearly dependent due to the following trig identity:

\[
\cos(3x) = -3 \cos x + 4 \cos^3 x
\]

Next, we choose two of the vectors, say \( \cos x \) and \( \cos(3x). \) These vectors are linearly independent (since they are not multiples of each other), so they form a basis for the subspace. Thus, the dimension of the subspace is 2.