Math 601 Solutions to Homework 11

1. Consider the curve given by the following parametric equations:
   \[ x = \cos(t) \quad \text{and} \quad y = \sin(2t) \]
   Let \( D \) be the region inside this curve and to the right of the \( y \)-axis:

   (a) Use Green’s Theorem to compute \( \iint_D x^2 \, dA \).

   (b) Use Green’s Theorem to find the area of the region \( D \).

Answer:

(a) We need to find a vector field \( \mathbf{F} \) such that \( \operatorname{rot}(\mathbf{F}) = x^2 \). The vector field \( \mathbf{F} = -yx^2 \mathbf{i} \) will work (as would \( \frac{1}{3}x^3 \mathbf{j} \) or lots of other possibilities). Then, by Green’s Theorem:

   \[
   \iint_D x^2 \, dA = \oint_{\partial D} -yx^2 \, dx
   \]

   From the parametrization of the curve, we have:

   \[
   x = \cos t \quad \quad dx = -\sin t \, dt
   \]

   \[
   y = \sin(2t) \quad \quad dy = 2 \cos(2t) \, dt
   \]

   The bounds for \( t \) are \(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\) (we can see this by noting that \( x \) and \( y \) are both 0 when \( t \) is a multiple of \( \pi/2 \), and then by plotting a few points of the parametric curve to make sure we choose the right-side of the lemniscate).

   We can now compute the integral (recall that \( \sin(2t) = 2\sin t \cos t \)):

   \[
   \iint_D x^2 \, dA = \oint_{\partial D} -yx^2 \, dx
   \]

   \[
   = \int_{-\pi/2}^{\pi/2} -\sin(2t) \cos^2 t (-\sin t) \, dt
   \]

   \[
   = \int_{-\pi/2}^{\pi/2} 2 \sin^2 t \cos^3 t \, dt
   \]

   \[
   = \int_{-\pi/2}^{\pi/2} 2 \sin^2 t (1 - \sin^2 t) \cos t \, dt
   \]
We use the substitution $u = \sin t$, $du = \cos t \, dt$:

\[
\int \int_D x^2 \, dA = \int_{-1}^{1} 2u^2(1 - u^2) \, du
\]

\[
= \left[ \frac{2}{3}u^3 - \frac{2}{5}u^5 \right]_{-1}^{1} = \frac{8}{15}
\]

**Note:** If we had used a different vector field, we would have obtained the same answer at the end.

(b) To compute the area, we need to evaluate the integral $\int_D dA$, so we need to find a vector field $\mathbf{F}$ such that $\text{rot}(\mathbf{F}) = 1$. The vector field $\mathbf{F} = -y \mathbf{i}$ will work (as would $x \mathbf{j}$ or lots of other possibilities). Then, by Green’s Theorem:

\[
\int \int_D dA = \oint_{\partial D} -y \, dx
\]

As in part (a), we have:

\[
x = \cos t \quad \text{dx} = -\sin t \, dt
\]

\[
y = \sin(2t) \quad \text{dy} = 2\cos(2t) \, dt
\]

with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

We can now compute the integral (recall that $\sin(2t) = 2\sin t \cos t$):

\[
\int \int_D dA = \oint_{\partial D} -y \, dx
\]

\[
= \int_{-\pi/2}^{\pi/2} -\sin(2t)(-\sin t) \, dt
\]

\[
= \int_{-\pi/2}^{\pi/2} 2\sin^2 t \cos t \, dt
\]

We use the substitution $u = \sin t$, $du = \cos t \, dt$:

\[
\int \int_D dA = \int_{-1}^{1} 2u^2 \, du
\]

\[
= \left[ \frac{2}{3}u^3 \right]_{-1}^{1} = \frac{4}{3}
\]
2. Consider the surface integral

$$\int \int_S z \, dA$$

where $S$ is the upper half ($z \geq 0$) of the sphere $x^2 + y^2 + z^2 = 1$.

(a) Set up this integral using a parametrization where $t = x$ and $u = y$.

(b) Set up this integral using a parametrization where $t = r$ and $u = \theta$.

(c) Set up this integral using a parametrization where $t = \theta$ and $u = \phi$.

**Answer:**

(a) With $t = x$ and $u = y$, we have the parametric equations:

\[
x = t, \quad \quad -\sqrt{1-u^2} \leq t \leq \sqrt{1-u^2}
\]
\[
y = u, \quad \quad -1 \leq u \leq 1
\]
\[
z = \sqrt{1-t^2-u^2}
\]

Then, the tangent vectors to the surface are

\[
T_t = \left(1, 0, \frac{-t}{\sqrt{1-t^2-u^2}}\right)
\]
\[
T_u = \left(0, 1, \frac{-u}{\sqrt{1-t^2-u^2}}\right)
\]
And, the normal vector to the surface is

\[
T_t \times T_u = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & -\frac{t}{\sqrt{1-t^2-u^2}} \\
0 & 1 & -\frac{u}{\sqrt{1-t^2-u^2}} \\
\end{vmatrix}
\]
\[
= \left(\frac{t}{\sqrt{1-t^2-u^2}}\right) \mathbf{i} + \left(\frac{u}{\sqrt{1-t^2-u^2}}\right) \mathbf{j} + \mathbf{k}
\]
Thus:

\[
dA = \|T_t \times T_u\| \, dt \, du = \sqrt{\frac{t^2}{1-t^2-u^2} + \frac{u^2}{1-t^2-u^2} + 1} \, dt \, du
\]
\[
= \sqrt{\frac{1}{1-t^2-u^2}} \, dt \, du
\]

Thus:

\[
\int \int_S z \, dA = \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \sqrt{1-t^2-u^2} \|T_t \times T_u\| \, dt \, du
\]
\[
= \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \frac{1}{\sqrt{1-t^2-u^2}} \sqrt{1-t^2-u^2} \, dt \, du
\]
\[
= \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dt \, du
\]
(b) With $t = r$ and $u = \theta$, we have the parametric equations:

\[
\begin{align*}
    x &= t \cos u & 0 \leq t \leq 1 \\
    y &= t \sin u \\
    z &= \sqrt{1 - t^2} & 0 \leq u \leq 2\pi
\end{align*}
\]

Then, the tangent vectors to the surface are

\[
T_t = \left( \cos u, \sin u, \frac{-t}{\sqrt{1-t^2}} \right)
\]

\[
T_u = (-t \sin u, t \cos u, 0)
\]

And, the normal vector to the surface is

\[
T_t \times T_u = \begin{vmatrix}
    i & j & k \\
    \cos u & \sin u & -t/\sqrt{1-t^2} \\
    -t \sin u & t \cos u & 0
\end{vmatrix}
\]

\[
= \left( \frac{t^2 \cos u}{\sqrt{1-t^2}} \right) i + \left( \frac{t^2 \sin u}{\sqrt{1-t^2}} \right) j + t k
\]

Thus:

\[
dA = \|T_t \times T_u\| \, dt \, du = \sqrt{\frac{t^4 \cos^2 u}{1-t^2} + \frac{t^4 \sin^2 u}{1-t^2} + t^2} \, dt \, du
\]

\[
= \sqrt{\frac{t^2}{1-t^2}} \, dt \, du
\]

Thus:

\[
\int \int_S z \, dA = \int_0^{2\pi} \int_0^1 \sqrt{1-t^2} \|T_t \times T_u\| \, dt \, du
\]

\[
= \int_0^{2\pi} \int_0^1 \sqrt{1-t^2} \sqrt{\frac{t^2}{1-t^2}} \, dt \, du
\]

\[
= \int_0^{\frac{\pi}{2}} \int_0^1 t \, dt \, du
\]

(c) Recall that $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$ and $z = \rho \cos \phi$. On the sphere, we have $\rho = 1$. Thus, if we use the parameters $t = \theta$ and $u = \phi$, we have the parametric equations:

\[
\begin{align*}
    x &= \sin u \cos t & 0 \leq t \leq 2\pi \\
    y &= \sin u \sin t \\
    z &= \cos u & 0 \leq u \leq \frac{\pi}{2}
\end{align*}
\]
Then, the tangent vectors to the surface are

\[ T_t = (-\sin u \sin t, \sin u \cos t, 0) \]

\[ T_u = (\cos u \cos t, \cos u \sin t, -\sin u) \]

And, the normal vector to the surface is

\[ T_t \times T_u = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  -\sin u \sin t & \sin u \cos t & 0 \\
  \cos u \cos t & \cos u \sin t & -\sin u \\
\end{vmatrix} \]

\[ = -\sin^2 u \cos t \mathbf{i} - \sin^2 u \sin t \mathbf{j} - \sin u \cos u \mathbf{k} \]

Thus:

\[ dA = \|T_t \times T_u\| \, dt \, du = \sqrt{\sin^4 u \cos^2 t + \sin^4 u \sin^2 t + \sin^2 u \cos^2 u} \, dt \, du \]

\[ = \sqrt{\sin^4 u + \sin^2 u \cos^u} \, dt \, du \]

\[ = \sqrt{\sin^2 u} \, dt \, du \]

Thus:

\[ \int \int_S z \, dA = \int_0^{\pi/2} \int_0^{2\pi} \cos u \|T_t \times T_u\| \, dt \, du \]

\[ = \int_0^{\pi/2} \int_0^{2\pi} \cos u \sin t \, dt \, du \]

\[ = \int_0^{\pi/2} \int_0^{2\pi} \cos u \sin u \, dt \, du \]

3. Use geometric reasoning to evaluate the following surface integrals:

(a) The integral

\[ \int \int_S (x^2 + y^2)^2 \, dA \]

where \( S \) is the surface defined by \( x^2 + y^2 = 4 \) for \( 2 \leq z \leq 5 \).

(b) The integral

\[ \int \int_S (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \cdot d\mathbf{A} \]

where \( S \) is the sphere of radius 3 centered at the origin.

(c) The integral

\[ \int \int_S (z \mathbf{k}) \cdot d\mathbf{A} \]

where \( S \) is the boundary of the region defined by \( x^2 + y^2 \leq 4 \) and \( 1 \leq z \leq 3 \).
(d) The integral
\[ \int \int_S (x^2 \mathbf{i} + y \mathbf{j} + 3z \mathbf{k}) \cdot d\mathbf{A} \]
where \( S \) is the boundary of the region defined by \( 0 \leq x \leq 3, 1 \leq y \leq 2, \) and \( 0 \leq z \leq 4. \)

**Answer:**
The surfaces in parts (b) through (d) should have been oriented. We will assume that all three surfaces are oriented with unit normal vectors pointing outwards. If you orient them the opposite way, the answer would be negated.

In general, closed surfaces are usually assumed to be oriented outwards, and closed loops are usually assumed to be oriented counterclockwise if no orientation is given.

(a) The surface is a cylinder about the \( z \)-axis with radius 2 between \( z = 2 \) and \( z = 5. \) On the surface, the value of \( (x^2 + y^2)^2 \) is \( 4^2 = 16. \) Thus:
\[ \int \int_S (x^2 + y^2)^2 \, dA = \int \int_S 16 \, dA = 16 \text{ (surface area of cylinder)} \]
The area of the cylinder is \( 2\pi rh = 2\pi(2)(5 - 2) = 12\pi. \) Thus:
\[ \int \int_S (x^2 + y^2)^2 \, dA = 16(12\pi) = 192\pi \]

(b) We want to write the integral as a scalar surface integral \( \int \int_S \mathbf{F} \cdot \mathbf{N} \, dA \)
where \( \mathbf{N} \) is the unit normal vector to the surface. Since \( S \) is a sphere centered at the origin, the vector \( x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) is normal to \( S \) and points in the outward direction. So:
\[ \mathbf{N} = \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} \]
On the surface \( S, \) \( x^2 + y^2 + z^2 = 9, \) so
\[ \mathbf{N} = \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{3} \]
Thus,
\[ \mathbf{F} \cdot \mathbf{N} = (x, y, z) \cdot \frac{(x, y, z)}{3} = \frac{x^2 + y^2 + z^2}{3} = 3 \]
Thus, we want to evaluate the integral
\[ \int \int_S 3 \, dA = 3 \text{ (surface area of sphere)} \]
The surface area of a sphere is \(4\pi r^2 = 4\pi (3)^2 = 36\pi\). Thus,

\[
\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{A} = 3(36\pi) = 108\pi
\]

**Note:** If we had oriented the sphere in the other direction, we would get \(-108\pi\).

(c) The region is a cylinder about the \(z\)-axis with radius 2 between \(z = 1\) and \(z = 3\). Since \(S\) is the boundary of this region, \(S\) includes the cylinder \(x^2 + y^2 = 4, 1 \leq z \leq 3\), and the top and bottom circles. We will need to compute the vector surface integral for each piece individually. Let us call these three pieces \(S_1\) (the cylinder), \(S_2\) (the top disc), and \(S_3\) (the bottom disc).

The normal vector to the cylinder \(x^2 + y^2 = 4\) is perpendicular to the vector field \(\mathbf{F} = z\mathbf{k}\). Thus, \(\mathbf{F} \cdot \mathbf{N} = 0\), so the surface integral is 0:

\[
\iint_{S_1} (z\mathbf{k}) \cdot d\mathbf{A} = 0
\]

The normal vector to the top disc points is \(\mathbf{N} = (0, 0, 1)\). Thus,

\[
\mathbf{F} \cdot \mathbf{N} = (0, 0, z) \cdot (0, 0, 1) = z
\]

Since the top disc is on the plane \(z = 3\), we have \(\mathbf{F} \cdot \mathbf{N} = 3\). Thus:

\[
\iint_{S_2} (z\mathbf{k}) \cdot d\mathbf{A} = \iint_{S_2} 3\, dA = 3(\text{area of disc})
\]

Since the area of the disc is \(\pi r^2 = \pi (2^2) = 4\pi\), we have:

\[
\iint_{S_2} (z\mathbf{k}) \cdot d\mathbf{A} = 3(4\pi) = 12\pi
\]

The normal vector to the bottom disc points is \(\mathbf{N} = (0, 0, -1)\). Thus,

\[
\mathbf{F} \cdot \mathbf{N} = (0, 0, z) \cdot (0, 0, -1) = -z
\]

Since the top disc is on the plane \(z = 1\), we have \(\mathbf{F} \cdot \mathbf{N} = -1\). Thus:

\[
\iint_{S_3} (z\mathbf{k}) \cdot d\mathbf{A} = \iint_{S_3} -1\, dA = -(\text{area of disc})
\]

Since the area of the disc is \(\pi r^2 = \pi (2^2) = 4\pi\), we have:

\[
\iint_{S_3} (z\mathbf{k}) \cdot d\mathbf{A} = -4\pi
\]
Our answer is the sum of these three results:

\[
\int\int_S (z \mathbf{k}) \cdot d\mathbf{A} = 12\pi - 4\pi = 8\pi
\]

**Note:** If we had oriented the surface in the other direction, we would get \(-8\pi\).

(d) The region is a box. We need to compute the surface integral on each of the 6 faces of the box.

On the bottom face, \(z = 0\), and \(\mathbf{N} = (0, 0, -1)\). So:

\[
\mathbf{F} \cdot \mathbf{N} = (x^2, y, 3z) \cdot (0, 0, -1) = -3z = 0
\]

Thus, the surface integral on the bottom face is 0.

On the top face, \(z = 4\) and \(\mathbf{N} = (0, 0, 1)\). So:

\[
\mathbf{F} \cdot \mathbf{N} = (x^2, y, 3z) \cdot (0, 0, 1) = 3z = 3(4) = 12
\]

Thus, the surface integral on the top face is 12(area of top face) = 12(1)(3) = 36.

On the face where \(x = 0\), we have \(\mathbf{N} = (-1, 0, 0)\). So:

\[
\mathbf{F} \cdot \mathbf{N} = (x^2, y, 3z) \cdot (-1, 0, 0) = -x^2 = 0
\]

Thus, the surface integral on this face is 0.

On the face where \(x = 3\), we have \(\mathbf{N} = (1, 0, 0)\). So:

\[
\mathbf{F} \cdot \mathbf{N} = (x^2, y, 3z) \cdot (1, 0, 0) = x^2 = 3^2 = 9
\]

Thus, the surface integral on this face is 9(area of face) = 9(1)(4) = 36.

On the face where \(y = 1\), we have \(\mathbf{N} = (0, -1, 0)\). So:

\[
\mathbf{F} \cdot \mathbf{N} = (x^2, y, 3z) \cdot (0, -1, 0) = -y = -1
\]

Thus, the surface integral on this face is \(-(area\ of\ face) = -3(4) = -12\).

On the face where \(y = 2\), we have \(\mathbf{N} = (0, 1, 0)\). So:

\[
\mathbf{F} \cdot \mathbf{N} = (x^2, y, 3z) \cdot (0, 1, 0) = y = 2
\]

Thus, the surface integral on this face is 2(area of face) = 2(3)(4) = 24.

The answer is the sum of all of these results:

\[
\int\int_S (x^2 \mathbf{i} + y \mathbf{j} + 3z \mathbf{k}) \cdot d\mathbf{A} = 0 + 36 + 0 + 36 - 12 + 24 = 84
\]

**Note:** If we had oriented the surface in the other direction, we would get \(-84\).
4. Let \( S \) be the surface \((r - 5)^2 + 9z^2 = 9\)

(a) Find parametric equations for the surface \( S \).

(b) Consider the vector field \( \mathbf{F}(x, y, z) = xz \mathbf{i} + yz \mathbf{j} - \frac{1}{z} \mathbf{k} \). Calculate the surface integral \( \iint_S \mathbf{F} \cdot d\mathbf{A} \).

Answer:

(a) If we look at the surface in the \( rz \)-plane, we have the ellipse

\[
\frac{(r - 5)^2}{9} + z^2 = 1
\]

We can parameterize this ellipse as:

\[
r = 5 + 3 \cos t \\
z = \sin t \\
0 \leq t \leq 2\pi
\]

The surface is obtained by rotating this ellipse about the \( z \)-axis. By letting \( \theta = u \), we can parameterize the surface in cylindrical coordinates:

\[
r = 5 + 3 \cos t \\
\theta = u \\
z = \sin t \\
0 \leq t \leq 2\pi \\
0 \leq u \leq 2\pi
\]

Since \( x = r \cos \theta \) and \( y = r \sin \theta \), we can parameterize the surface in Cartesian coordinates:

\[
x = (5 + 3 \cos t) \cos u \quad 0 \leq t \leq 2\pi \\
y = (5 + 3 \cos t) \sin u \quad 0 \leq u \leq 2\pi \\
z = \sin t
\]

(b) As in problem 3, the surface should have been given an orientation. We will assume that the surface is oriented with unit vectors pointing away from the bounded region. (The surface is the surface of a torus or doughnut, so the normal vectors are pointing away from the inside of the torus.)

Using the above parametrization, we compute the tangent vectors to the surface, \( T_t \) and \( T_u \):

\[
T_t = (-3 \sin t \cos u, -3 \sin t \sin u, \cos t) \\
T_u = (- \sin u(5 + 3 \cos t), \cos u(5 + 3 \cos t), 0)
\]
Next, we compute $T_t \times T_u$:

$$
T_t \times T_u = \begin{vmatrix}
i & j & k \\
-3 \sin t \cos u & -3 \sin t \sin u & \cos t \\
- \sin u(5 + 3 \cos t) & \cos u(5 + 3 \cos t) & 0
\end{vmatrix}
$$

$$
= - \cos t \cos u(5 + 3 \cos t) i - \cos t \sin u(5 + 3 \cos t) j \\
+ (5 + 3 \cos t) \left( -3 \sin t \cos^2 u - 3 \sin t \sin^2 u \right) k
$$

$$
= (5 + 3 \cos t) \left( - \cos t \cos u i - \cos t \sin u j - 3 \sin t k \right)
$$

Now, we need to check that we have oriented the surface correctly. We can do this by checking the direction of the normal vector $T_t \times T_u$ at a point on the surface.

For simplicity, we choose the point where $t = 0$ and $u = 0$. Plugging $t = 0$ and $u = 0$ into the parametric equations, we get the point $(8, 0, 0)$. Plugging $t = 0$ and $u = 0$ into the formula for $T_t \times T_u$, we see that the normal vector at this point is $(-8, 0, 0)$.

This vector points into the torus instead of out of the torus, so we will need to negate the normal vector.

We compute $(xz \mathbf{i} + yz \mathbf{j} - \frac{1}{z} \mathbf{k}) \cdot (-T_t \times T_u)$:

$$
(xz \mathbf{i} + yz \mathbf{j} - \frac{1}{z} \mathbf{k}) \cdot (-T_t \times T_u) = \cos t \sin t \cos^2 u(5 + 3 \cos t)^2 \\
+ \cos t \sin t \sin^2 u(5 + 3 \cos t)^2 \\
-3(5 + 3 \cos t)
$$

$$
= \cos t \sin t(5 + 3 \cos t)^2 - 3(5 + 3 \cos t)
$$

Thus:

$$
\iint_S \mathbf{F} \cdot d\mathbf{A} = \int_0^{2\pi} \int_0^{2\pi} \left( \cos t \sin t(5 + 3 \cos t)^2 - 3(5 + 3 \cos t) \right) \, du \, dt
$$

$$
= \int_0^{2\pi} 2\pi \left( \cos t \sin t(5 + 3 \cos t)^2 - 3(5 + 3 \cos t) \right) \, dt
$$

$$
= 2\pi \int_0^{2\pi} \cos t \sin t(5 + 3 \cos t)^2 \, dt - 6\pi \int_0^{2\pi} (5 + 3 \cos t) \, dt
$$

On the first integral, we use the substitution $v = \cos t$, $dv = - \sin t \, dt$.

Then:

$$
\iint_S \mathbf{F} \cdot d\mathbf{A} = 2\pi \int_1^{-1} -v(5 + 3v)^2 \, dv - 6\pi \int_0^{2\pi} (5 + 3 \cos t) \, dt
$$

$$
= 0 - 60\pi^2 = -60\pi^2
$$

Note: If we had oriented the surface the other direction, we would get $60\pi^2$. 