Math 601 Solutions to Homework 10

1. Consider the vector field \( \mathbf{F}(x, y) = (2x + 2y)i + (2x - y)j \).

   (a) Find the parametric equations for the flow line of \( \mathbf{F} \) beginning at the point \((0, 5)\).

   (b) Find \( \text{div}(\mathbf{F}) \).

   (c) Find \( \text{rot}(\mathbf{F}) \).

Answer:

(a) We need to solve the system of differential equations:

\[
\frac{dx}{dt} = 2x + 2y \\
\frac{dy}{dt} = 2x - y
\]

This is a system of linear differential equations, which we can write as:

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

To solve this system, we need to compute the eigenvalues and eigenvectors of the matrix.

We can quickly find the eigenvalues if we remember that the product of the eigenvalues must equal the determinant of the matrix, and the sum of the eigenvalues must equal the trace of the matrix.

So, \( \lambda_1 \lambda_2 = -6 \) and \( \lambda_1 + \lambda_2 = 1 \). By inspection we see that the eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = -2 \). (This method only works for \( 2 \times 2 \) matrices.)

Now, we find the eigenvectors associated with each eigenvalue. For \( \lambda_1 = 3 \), we have:

\[
\text{nullspace} \begin{bmatrix} \lambda_1 - 2 & -2 \\ -2 & \lambda_1 + 1 \end{bmatrix} = \text{nullspace} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}
\]

\[
= \text{nullspace} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}
\]

\[
= \text{Span} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)
\]

Thus, the vector \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) is an eigenvector associated with the eigenvalue \( \lambda_1 = 3 \).
For $\lambda_2 = -2$, we have:

$$\text{nullspace} \begin{bmatrix} \lambda_2 - 2 & -2 \\ -2 & \lambda_2 + 1 \end{bmatrix} = \text{nullspace} \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$$

$$= \text{nullspace} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \text{Span} \left( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

Thus, the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = -2$.

Thus, the general solution to the system of linear differential equations is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We would like to find the solution which begins at the point $(0, 5)$, so we want $x = 0, y = 5$ when $t = 0$. We can plug those numbers into the above equation and solve for $c_1$ and $c_2$. We get the equations:

$$0 = 2c_1 + c_2$$
$$5 = c_1 - 2c_2$$

Solving for $c_1$ and $c_2$, we get $c_1 = 1$ and $c_2 = -2$. Thus, the parametric equation for the flow line is:

$$\begin{align*}
x &= 2e^{3t} - 2e^{-2t} \\
y &= e^{3t} + 4e^{-2t}
\end{align*}$$

(b) $\text{div}(\mathbf{F}) = \frac{\partial}{\partial x} (2x + 2y) + \frac{\partial}{\partial y} (2x - y) = 2 - 1 = 1$

(c) $\text{rot}(\mathbf{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2x + 2y & 2x - y \end{vmatrix} = 2 - 2 = 0$
2. Find a vector field \( \mathbf{F} \) whose flow lines are the parametric curves
\[
x = t \quad y = C(t^2 + 1)
\]

**Answer:** The components of the vector field \( \mathbf{F} \) must be \( F_x = \frac{dx}{dt} = 1 \) and \( F_y = \frac{dy}{dt} = 2Ct \). We just need to write these in terms of \( x \) and \( y \). Using the equations for the flow lines, we know that \( t = x \). Also, the equation \( y = C(t^2 + 1) \) gives us \( C = \frac{y}{t^2 + 1} = \frac{y}{x^2 + 1} \). Thus, the vector field \( \mathbf{F} \) is:

\[
\mathbf{F}(x, y) = \mathbf{i} + \frac{2xy}{x^2 + 1} \mathbf{j}
\]

3. (a) Let \( C \) be the portion of the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) in the first quadrant. Evaluate the following integral:
\[
\int_C xy \, ds
\]

(b) Let \( C \) be the curve in \( \mathbb{R}^3 \) described by the equations \( \rho = \theta = \frac{\pi}{4} \) with endpoints \((x, y, z) = (0, 0, 0)\) and \((x, y, z) = (\pi\sqrt{2}, 0, \pi\sqrt{2})\). Evaluate the following integral:
\[
\int_C \frac{x}{z} \, dx + \frac{y}{z} \, dy + z^3 \, dz
\]

**Answer:**

(a) First, we parameterize the portion of the ellipse:
\[
x = 2 \cos t \\
y = 3 \sin t \\
0 \leq t \leq \frac{\pi}{2}
\]

Thus:
\[
dx = -2 \sin t \, dt \\
dy = 3 \cos t \, dt
\]

Thus:
\[
ds = \sqrt{dx^2 + dy^2} = \sqrt{4 \sin^2 t + 9 \cos^2 t} \, dt = \sqrt{4 + 5 \cos^2 t} \, dt
\]

Thus, the line integral is:
\[
\int_C xy \, ds = \int_0^{\pi/2} 6 \cos t \sin t \sqrt{4 + 5 \cos^2 t} \, dt
\]
We can evaluate this integral using the substitution \( u = 4 + 5 \cos^2 t \), \( du = -10 \cos t \sin t \). Thus:

\[
\int_C xy \, ds = \int_0^{\pi/2} 6 \cos t \sin t \sqrt{4 + 5 \cos^2 t} \, dt = \int_0^4 -\frac{3}{5} \sqrt{u} \, du = \left[ -\frac{2}{5} u^{3/2} \right]_0^4 = \frac{38}{5}
\]

(b) First we parameterize the curve \( C \) using \( t = \theta \). Then:

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta = \frac{t}{\sqrt{2}} \cos t \\
y &= \rho \sin \phi \sin \theta = \frac{t}{\sqrt{2}} \sin t \\
z &= \rho \cos \phi = \frac{t}{\sqrt{2}}
\end{align*}
\]

The bounds for \( t \) are chosen so that at \( t = 0 \), \((x, y, z) = (0, 0, 0)\) and at \( t = 2\pi \), \((x, y, z) = (\pi \sqrt{2}, 0, \pi \sqrt{2})\). Thus:

\[
\begin{align*}
dx &= \left( \frac{1}{\sqrt{2}} \cos t - \frac{t}{\sqrt{2}} \sin t \right) \, dt \\
dy &= \left( \frac{1}{\sqrt{2}} \sin t + \frac{t}{\sqrt{2}} \cos t \right) \, dt \\
dz &= \frac{1}{\sqrt{2}} \, dt
\end{align*}
\]

Thus, the integral becomes:

\[
\begin{align*}
\int_C \frac{x}{z} \, dx + \frac{y}{z} \, dy + z^3 \, dz &= \int_0^{2\pi} \left( \frac{t \cos t / \sqrt{2}}{t / \sqrt{2}} \left( \frac{1}{\sqrt{2}} \cos t - \frac{t}{\sqrt{2}} \sin t \right) + \frac{t \sin t / \sqrt{2}}{t / \sqrt{2}} \left( \frac{1}{\sqrt{2}} \sin t + \frac{t}{\sqrt{2}} \cos t \right) + \frac{t^3}{2 \sqrt{2}} \left( \frac{1}{\sqrt{2}} \right) \right) \, dt \\
\end{align*}
\]

This simplifies to:

\[
\int_C \frac{x}{z} \, dx + \frac{y}{z} \, dy + z^3 \, dz = \int_0^{2\pi} \left( \frac{1}{\sqrt{2}} + \frac{t^3}{4} \right) \, dt = \pi \sqrt{2} + \pi^4
\]
4. Use geometric reasoning to evaluate the following line integrals:

(a) The integral
\[ \int_C \left( x^2 + y^2 \right)^2 \, ds \]
where \( C \) is the circle of radius 3 centered at the origin.

(b) The integral
\[ \int_C -ry \, dx + rx \, dy \]
where \( C \) is the circle of radius 2 centered at the origin.

(c) The integral
\[ \int_C x \, dx + y \, dy + z \, dz \]
where \( C \) is any curve on the sphere \( x^2 + y^2 + z^2 = 9 \).

Answer:

(a) On the circle \( C \), the value of \( (x^2 + y^2)^2 \) is 81. If we integrate this function over the circle, we get 81 times the circumference of the circle, which is \( 81 \times 3 \times 2 \times \pi = 486 \pi \)

(b) The problem should have specified the orientation of \( C \). We will assume that \( C \) is oriented counterclockwise.

We want to write the integral as a scalar line integral \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \)
where \( \mathbf{T} \) is the unit tangent vector to the curve. Since \( C \) is a circle centered at the origin, the vector \( -yi + xj \) is tangent to \( C \) and points in the counterclockwise direction. So:
\[ \mathbf{T} = \frac{-yi + xj}{\sqrt{x^2 + y^2}} = \frac{-yi + xj}{r} \]

Thus, \( \mathbf{F} \cdot \mathbf{T} = (-ry, rx) \cdot \left( \frac{-y}{r}, \frac{x}{r} \right) = y^2 + x^2 \), so on the circle \( C \), \( \mathbf{F} \cdot \mathbf{T} = r^2 = 4 \). If we integrate this function over the circle, we get 4 times the circumference of the circle, which is \( 4 \times 2 \times \pi = 16 \pi \)

Note: If we orient \( C \) clockwise, we obtain \( -16 \pi \).

(c) The vector field \( \mathbf{F}(x, y, z) = xi + yj + zk \) points radially outwards, so the tangent vectors to the curve \( C \) are perpendicular to \( \mathbf{F} \).

Thus, \( \mathbf{F} \cdot \mathbf{T} = 0 \), so \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds = 0 \)
5. Consider the vector field \( \mathbf{F} = (2x + y) \mathbf{i} + (x + 3y^2) \mathbf{j} \).

(a) Show that \( \mathbf{F} \) is conservative.
(b) Find a function \( f \) such that \( \nabla f = \mathbf{F} \).
(c) Evaluate the integral \( \int_C \mathbf{F} \cdot d\mathbf{s} \) where \( C \) is the curve \( y = \sin(x^2) \) from \((0,0)\) to \((\sqrt{\pi},0)\).

**Answer:**

(a) We can show that \( \mathbf{F} \) is conservative by showing that \( \text{rot}(\mathbf{F}) = 0 \):

\[
\text{rot}(\mathbf{F}) = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
2x + y & x + 3y^2
\end{vmatrix} = 1 - 1 = 0
\]

Since \( \text{rot}(\mathbf{F}) = 0 \), \( \mathbf{F} \) is conservative.

(b) We want a function \( f \) such that

\[
\frac{\partial f}{\partial x} = 2x + y \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 3y^2
\]

From the first equation, we see that

\[
f(x, y) = x^2 + xy + g(y)
\]

where \( g(y) \) is some function of \( y \). If we differentiate this equation for \( f \) with respect to \( y \), we obtain

\[
\frac{\partial f}{\partial y} = x + \frac{dg}{dy}
\]

Comparing with the previous expression for \( \frac{\partial f}{\partial y} \), we see that \( \frac{dg}{dy} = 3y^2 \). Thus:

\[
g(y) = y^3 + C
\]

Thus:

\[
f(x, y) = x^2 + xy + y^3 + C
\]

(c) It is now easy to evaluate the line integral:

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \left[ x^2 + xy + y^3 \right]_{(\sqrt{\pi},0)}^{(0,0)} = \pi
\]
6. Let $C$ be the circle of radius 1 centered at the origin and oriented counterclockwise. Use Green’s Theorem to evaluate the following integral:

$$
\oint_C (e^y - y^3) \, dx + (xe^y + x^3) \, dy
$$

**Answer:** First, we compute the rotation of the vector field:

$$
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
 e^y - y^3 & xe^y + x^3 \\
\end{vmatrix} = e^y + 3x^2 - (e^y - 3y^2) = 3x^2 + 3y^2
$$

Thus, by Green’s Theorem:

$$
\oint_C (e^y - y^3) \, dx + (xe^y + x^3) \, dy = \iint_D (3x^2 + 3y^2) \, dA
$$

where $D$ is the region inside the circle of radius 1 centered at the origin. This integral will be easiest to compute using polar coordinates:

$$
\iint_D (3x^2 + 3y^2) \, dA = \int_0^{2\pi} \int_0^1 3r^3 \, dr \, d\theta = \frac{3\pi}{2}
$$