Math 601 Solutions to Homework 1

1. Find all solutions to the following system of linear equations:

\[
\begin{align*}
    x_2 + 2x_3 + x_4 &= 2 \\
    x_1 + x_2 + x_3 + x_4 &= 3 \\
    -2x_1 + x_2 + 4x_3 + x_4 &= 0 \\
    -x_1 + 2x_2 + 5x_3 + 2x_4 &= 3
\end{align*}
\]

Answer:
First, we write the system of linear equations as an augmented matrix:

\[
\begin{pmatrix}
    0 & 1 & 2 & 1 & 2 \\
    1 & 1 & 1 & 1 & 3 \\
    -2 & 1 & 4 & 1 & 0 \\
    -1 & 2 & 5 & 2 & 3
\end{pmatrix}
\]

Then, we perform row reductions to get the matrix in reduced row echelon form. The first row operation we perform is switching the first and second row.

\[
\begin{pmatrix}
    1 & 1 & 1 & 1 & 3 \\
    0 & 1 & 2 & 1 & 2 \\
    -2 & 1 & 4 & 1 & 0 \\
    -1 & 2 & 5 & 2 & 3
\end{pmatrix}
\] → \[
\begin{pmatrix}
    1 & 0 & 1 & 0 & 1 \\
    0 & 1 & 2 & 1 & 2 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}
\] → \[
\begin{pmatrix}
    1 & 1 & 1 & 1 & 3 \\
    0 & 1 & 2 & 1 & 2 \\
    0 & 3 & 6 & 3 & 6 \\
    0 & 3 & 6 & 3 & 6
\end{pmatrix}
\] → \[
\begin{pmatrix}
    1 & 0 & -1 & 0 & 1 \\
    0 & 1 & 2 & 1 & 2 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Writing this in equation form, we get:

\[
\begin{align*}
    x_1 - x_3 &= 1 \\
    x_2 + 2x_3 + x_4 &= 2
\end{align*}
\]
Solving for $x_1$ and $x_2$:

\[
\begin{align*}
    x_1 &= 1 + x_3 \\
    x_2 &= 2 - 2x_3 - x_4
\end{align*}
\]

Thus, for all real numbers $s$ and $t$,

\[
\begin{align*}
    x_1 &= 1 + s \\
    x_2 &= 2 - 2s - t \\
    x_3 &= s \\
    x_4 &= t
\end{align*}
\]

is a solution to the system of linear equations.

2. Balance the following chemical equation (see Application 3 on page 24 of the text for information on balancing chemical equations):

\[x_1\text{NO}_2 + x_2\text{H}_2\text{O} \rightarrow x_3\text{HNO}_2 + x_4\text{HNO}_3\]

**Answer:**

For each element involved in the chemical reaction, there should be the same amount involved on the right side of the equation and on the left side of the equation. Thus, for each element (N, O, and H), we get a linear equation:

\[
\begin{align*}
    \text{N:} & \quad x_1 = x_3 + x_4 \\
    \text{O:} & \quad 2x_1 + x_2 = 2x_3 + 3x_4 \\
    \text{H:} & \quad 2x_2 = x_3 + x_4
\end{align*}
\]

We can rewrite these equations with all of the variables on the left side:
\[
\begin{align*}
x_1 - x_3 - x_4 &= 0 \\
2x_1 + x_2 - 2x_3 - 3x_4 &= 0 \\
2x_2 - x_3 - x_4 &= 0
\end{align*}
\]

We can write these equations as an augmented matrix and row reduce:

\[
\begin{pmatrix}
1 & 0 & -1 & -1 & 0 \\
2 & 1 & -2 & -3 & 0 \\
0 & 2 & -1 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & -2 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0
\end{pmatrix}
\]

This gives us

\[
x_1 = 2x_4 \\
x_2 = x_4 \\
x_3 = x_4
\]

So, for all real numbers \( t \)

\[
x_1 = 2t \\
x_2 = t \\
x_3 = t \\
x_4 = t
\]

is a solution to the system of linear equations.

Since we just want to balance the chemical equation, we want one solution, with positive integer values for the variables. Also, for simplicity, it is best to use the solution with smallest integer values for the variables. If we let \( t = 1 \), we get the solution \( x_1 = 2, x_2 = 1, x_3 = 1 \) and \( x_4 = 1 \). Thus, our balanced chemical equation is

\[
2\text{NO}_2 + \text{H}_2\text{O} \longrightarrow \text{HNO}_2 + \text{HNO}_3
\]
3. Determine the amount of each current for the following network (see Application 4 on page 22 of the text for information on electrical networks):

\[ \text{Answer: Using Kirchoff’s first law (at each node, the incoming current equals the outgoing current), we get an equation for each node:} \]

\[ A: \quad i_1 = i_2 + i_5 \]
\[ B: \quad i_2 + i_4 = i_3 \]
\[ C: \quad i_3 + i_6 = i_2 \]
\[ D: \quad i_5 = i_4 + i_6 \]

Using Kirchoff’s second law (around every closed loop the total change in voltage is zero), we get an equation for every loop:

\[ \text{top cycle:} \quad 9 = 3i_2 + i_3 \]
\[ \text{lower left cycle:} \quad 0 = i_5 + 3i_4 - 3i_2 \]
\[ \text{lower right cycle:} \quad 0 = 3i_6 - i_3 - 3i_4 \]

We can rewrite these equations as an augmented matrix:
This matrix row reduces to

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Thus, the solution is

\[
i_1 = 5 \\
i_2 = 2 \\
i_3 = 3 \\
i_4 = 1 \\
i_5 = 3 \\
i_6 = 2
\]

4. Consider a linear system whose augmented matrix is of the form

\[
\begin{pmatrix}
1 & 2 & -6 & | & 4 \\
-3 & 2 & 2 & | & -4 \\
4 & -3 & a & | & b \\
\end{pmatrix}
\]
(a) For what values of $a$ and $b$ will the system have infinitely many solutions?

(b) For what values of $a$ and $b$ will the system be inconsistent?

(c) For what values of $a$ and $b$ will the system have exactly one solution?

**Answer:**

First, we row reduce the matrix. We get

$$
\begin{pmatrix}
1 & 2 & -6 & 4 \\
0 & 1 & -2 & 1 \\
0 & 0 & a+2 & b-5 \\
\end{pmatrix}
$$

Now, we can answer the questions:

(a) If $a = -2$ and $b = 5$, the system has infinitely many solutions.

(b) If $a = -2$ and $b \neq 5$ the system is inconsistent.

(c) If $a \neq -2$, then the system will have exactly one solution (in this case, it does not matter what $b$ equals).