Math 601 Solutions to Practice Problems for Test 3

1. The vectors \( (3, 2, 1) - (3, 1, 0) = (0, 1, 1) \) and \( (4, 1, 2) - (3, 1, 0) = (1, 0, 2) \) are both parallel to the plane, so the normal vector is the cross product:

\[
(0, 1, 1) \times (1, 0, 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = (2, 1, -1)
\]

Therefore, the equation of the plane is \( 2x + y - z = 7 \).

2. The vector \( (1, 1, 0) \times (3, 2, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = (-1, -1, -1) \) is perpendicular to both lines. The first line is contained in the plane \( x - y - z = -3 \), and the second line is contained in the parallel plane \( x - y - z = -1 \). The distance between these two lines is equal to the distance between these planes:

\[
\frac{(-1) - (-3)}{\|(-1, -1, -1)\|} = \frac{2}{\sqrt{3}}
\]

3. There are two possible ways to set up this integral:

\[
\int_0^{\sqrt{2}} \int_{x^2}^2 x \cos(y^2) \, dy \, dx \quad \text{or} \quad \int_0^{\sqrt{2}} \int_0^{\sqrt{y}} x \cos(y^2) \, dx \, dy
\]

The first integral is unhelpful, but the second can be integrated using \( u \)-substitution:

\[
\int_0^{\sqrt{2}} \int_0^{\sqrt{y}} x \cos(y^2) \, dx \, dy = \int_0^{\sqrt{2}} \left[ \frac{1}{2} x^2 \cos(y^2) \right]_{x=0}^{\sqrt{y}} \, dy
\]

\[
= \frac{1}{2} \int_0^{\sqrt{2}} y \cos(y^2) \, dy = \frac{1}{2} \left[ \frac{1}{2} \sin(y^2) \right]_0^{\sqrt{2}} = \frac{1}{4} \sin(4)
\]
4. The $z$-coordinate is the ratio $\frac{\iiint_R z \, dV}{\iiint_R dV}$. This can be evaluated as follows:

\[
\frac{\int_0^1 \int_0^{1-y} \int_0^{1-z} z \, dx \, dy \, dz}{\int_0^1 \int_0^{1-y} \int_0^{1-z} dx \, dy \, dz} = \frac{\int_0^1 \int_0^{1-y} z(1-z) \, dz \, dy}{\int_0^1 \int_0^{1-y} (1-z) \, dz \, dy} = \frac{\int_0^1 \left[ \frac{1}{2} z^2 - \frac{1}{3} z^3 \right]_{z=0}^{1-z} \, dy}{\int_0^1 \left[ z - \frac{1}{2} z^2 \right]_{z=0}^{1-z} \, dy} = \frac{\int_0^1 \left( \frac{1}{2} (1-y)^2 - \frac{1}{3} (1-y)^3 \right) \, dy}{\int_0^1 \left( 1-y - \frac{1}{2} (1-y)^2 \right) \, dy} = \frac{1}{6} - \frac{1}{12} = \frac{1}{4}
\]

5. We use spherical coordinates:

\[
\iiint_K \frac{1}{(x^2 + y^2 + z^2)^2} \, dV = \iiint_K \frac{1}{\rho^4} \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2} \sec \varphi} \frac{1}{\rho^3} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta
\]

\[
= \int_0^{2\pi} d\theta \int_0^{\pi/4} \int_0^{\sqrt{2} \sec \varphi} \frac{\sin \varphi}{\rho^2} \, d\rho \, d\varphi
= 2\pi \int_0^{\pi/4} \left[ -\frac{\sin \varphi}{\rho} \right]_{\rho=\sqrt{2} \sec \varphi}^{\rho=\rho} \, d\varphi
= 2\pi \int_0^{\pi/4} \left( -\frac{1}{2} \sin \varphi + \frac{1}{\sqrt{2}} \sin \varphi \cos \varphi \right) \, d\varphi
= 2\pi \left[ \frac{1}{2} \cos \varphi + \frac{1}{2\sqrt{2}} \sin^2 \varphi \right]_{\varphi=0}^{\pi/4}
= 2\pi \left[ \frac{1}{2} \left( \frac{1}{\sqrt{2}} - 1 \right) + \frac{1}{4 \sqrt{2}} \right] = \left( \frac{3}{4 \sqrt{2}} - 1 \right) \pi
\]

6. We use cylindrical coordinates:

\[
\iiint_K \frac{1}{z^2 e^{x^2+y^2}} \, dV = \iiint_K \frac{1}{z^2 e^{r^2}} \, dV = \int_0^{\pi/2} \int_0^\infty \int_0^\infty \frac{1}{z^2 e^{r^2}} r \, dz \, dr \, d\theta
\]

\[
= \int_0^{\pi/2} d\theta \int_0^\infty r e^{-r^2} \, dr \int_5^\infty \frac{1}{z^2} \, dz
= \frac{\pi}{2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^\infty \left[ -\frac{1}{z} \right]_5^\infty
= \left( \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{5} \right) = \frac{\pi}{20}
\]
7. The Jacobian of the transformation is:

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\
\frac{\partial y}{\partial t} & \frac{\partial y}{\partial u}
\end{vmatrix} = \begin{vmatrix}
3e^{-u} & -3te^{-u} \\
e^u & te^u
\end{vmatrix} = 6t
\]

Therefore:

\[
\int_\pi y\,dA = \int_1^2\int_0^1 y|J|\,du\,dt = \int_1^2\int_0^1 (te^u)(6t)\,du\,dt
\]

\[
= \int_1^2 6t^2\,dt\int_0^1 e^u\,du = 14(e - 1)
\]

8. Using the coordinates \( u = 2x - y \) and \( v = -x + 2y \) (so \( x = \frac{2}{3}u + \frac{1}{3}v \) and \( y = \frac{1}{3}u + \frac{2}{3}v \)), the parallelogram has vertices \((0, 0), (0, 3), (3, 0),\) and \((3, 3)\). The Jacobian of the transformation is:

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \begin{vmatrix}
2/3 & 1/3 \\
1/3 & 2/3
\end{vmatrix} = \frac{1}{3}
\]

Therefore:

\[
\int_\pi \sqrt{2x - y} \,dA = \int_\pi \sqrt{u} \,dA = \int_0^3 \int_0^3 \sqrt{u} \,1/3 \,du\,dv
\]

\[
= \frac{1}{3} \int_0^3 u \,du \int_0^3 \frac{1}{\sqrt{1 + v}} \,dv = \frac{1}{3} \left[ \frac{2}{3}u^{3/2} \right]_0^3 \left[ 2\sqrt{1 + v} \right]_0^3
\]

\[
= \left( \frac{1}{3} \right) \left( \frac{2}{3} \cdot 3\sqrt{3} \right) \left( 2\sqrt{4} - 2\sqrt{1} \right) = \frac{4}{3}\sqrt{3}
\]

9. The first surface is a level surface of the function \( f(x, y, z) = x^3 + yz \), while the second surface is a level curve of the function \( g(x, y, z) = 5xy^2 - 2yz \). Therefore, the gradients of these functions should be normal to the surfaces. At the point \((1, 2, 3)\):

\[
\nabla f = (3x^2, z, y) = (3, 3, 2) \quad \text{and} \quad \nabla g = (5y^2, 10xy - 2z, -2y) = (20, 14, -4)
\]

The angle between these vectors is:

\[
\cos \theta = \frac{(3, 3, 2) \cdot (20, 14, -4)}{\|(3, 3, 2)\| \|(20, 14, -4)\|} = \frac{94}{\sqrt{22}\sqrt{612}}
\]

Therefore, \( \theta = \cos^{-1}\left( \frac{94}{\sqrt{22}\sqrt{612}} \right) \).
10.  
(a) It looks like $F_x$ is constant, and $\frac{\partial F_y}{\partial y} > 0$, so the divergence is greater than zero.
(b) This vector field appears to be constant, so the divergence is equal to zero.
(c) It looks like $F_x$ is constant, and $\frac{\partial F_y}{\partial y} < 0$, so the divergence is less than zero.
(d) It looks like $F_x$ is always zero, and $\frac{\partial F_y}{\partial y} > 0$, so the divergence is greater than zero.

11.  
(a) The flow lines satisfy the differential equations:

$$\frac{dx}{dt} = x \quad \text{and} \quad \frac{dy}{dt} = xy^2$$

The solution to the first equation is $x = Ae^t$. Plugging this into the second equation gives:

$$\frac{dy}{dt} = Ae^t y^2, \quad \text{so} \quad \frac{dy}{y^2} = Ae^t dt \quad \text{so} \quad -\frac{1}{y} = Ae^t + B$$

Therefore, the flow lines are the curves

$$x = Ae^t \quad \text{and} \quad y = -\frac{1}{Ae^t + B}.$$  

(b) $\text{div}(\mathbf{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial}{\partial x} [x] + \frac{\partial}{\partial y} [xy^2] = 1 + 2xy$.

(c) $\text{rot}(\mathbf{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_x & F_y \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & xy^2 \end{vmatrix} = y^2 - 0 = y^2.$
12. The flow lines satisfy the differential equations:

\[ \frac{dx}{dt} = \frac{x - y}{x^2 + y^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{x + y}{x^2 + y^2} \]

We change to polar coordinates:

\[ \frac{dr}{dt} = \frac{\partial r}{\partial x} \frac{dx}{dt} + \frac{\partial r}{\partial y} \frac{dy}{dt} = \left( \frac{x}{r} \right) \left( \frac{x - y}{r^2} \right) + \left( \frac{y}{r} \right) \left( \frac{x + y}{r^2} \right) = \frac{x^2 + y^2}{r^3} = \frac{1}{r} \]

and:

\[ \frac{d\theta}{dt} = \frac{\partial \theta}{\partial x} \frac{dx}{dt} + \frac{\partial \theta}{\partial y} \frac{dy}{dt} = -\left( \frac{y}{r^2} \right) \left( \frac{x - y}{r^2} \right) + \left( \frac{x}{r^2} \right) \left( \frac{x + y}{r^2} \right) = \frac{y^2 + x^2}{r^4} = \frac{1}{r^2} \]

In summary:

\[ \frac{dr}{dt} = \frac{1}{r} \quad \text{and} \quad \frac{d\theta}{dt} = \frac{1}{r^2} \]

We can solve the first equation:

\[ r \frac{dr}{dt} = dt, \quad \text{so} \quad \frac{1}{2} r^2 = t + A, \quad \text{so} \quad r = \sqrt{2t + 2A} \]

Since \( r = 1 \) when \( t = 0 \), the constant \( 2A \) must be 1, so \( r = \sqrt{2t + 1} \). Plugging this into the equation for \( \theta \) gives:

\[ \frac{d\theta}{dt} = \frac{1}{2t + 1}, \quad \text{so} \quad \theta = \int \frac{dt}{2t + 1} = \frac{1}{2} \ln(2t + 1) + B \]

Since \( \theta = 0 \) when \( t = 0 \), the constant \( B \) must be zero. So:

\[ r = \sqrt{2t + 1} \quad \text{and} \quad \theta = \frac{1}{2} \ln(2t + 1) = \ln \sqrt{2t + 1} \]

Plugging this into the equations \( x = r \cos \theta \) and \( y = r \sin \theta \) gives the flow line:

\[ x = \sqrt{2t + 1} \cos \left( \ln \sqrt{2t + 1} \right) \quad \text{and} \quad y = \sqrt{2t + 1} \sin \left( \ln \sqrt{2t + 1} \right) \]