Math 308 Week 6 Solutions

There is a solution manual to Chapter 4 online: www.pearsoncustom.com/tamu_math/. This online solutions manual contains solutions to some of the suggested problems. Here are solutions to suggested problems that cannot be found in the online solutions manual.

Matlab 10

23. Find the solution to the following initial value problem:
\[ t^2y'' = (y')^2, \quad y(1) = 3, \quad y'(1) = 2 \]

Answer: We enter into Matlab:

\[
\text{>> dsolve('t^2*D2y = (Dy)^2', 'y(1) = 3', 'Dy(1) = 2', 't')}
\]

We get:
\[
y = -2t - 4 \ln \left(-1 + \frac{t}{2}\right) + 5 - 4 \ln 2 + 4\pi i
\]

24. Find the solution to the following initial value problem:
\[ y'' = yy', \quad y(0) = 0, \quad y'(0) = 2 \]

Answer: If we enter the differential equation with both initial conditions into Matlab, Matlab refuses to find a solution. It will find a solution, though, if we just give it one of the initial conditions. And then, we can use the second initial condition to find the solution. Here is one list of commands that will determine the constant \(C1\):

\[
\text{>> f = dsolve('D2y = y*Dy', 'y(0) = 0', 't')}
\]

\[
\text{>> g = diff(f, 't')}
\]

\[
\text{>> h = subs(g, 't', 0)}
\]

\[
\text{>> C1 = solve(g - 2, 'C1')}
\]

We get that \(C1 = 1/\sqrt{2}\) or \(C1 = -1/\sqrt{2}\). Substitute either of them in for \(C1\):

\[
\text{>> subs(f, 'C1', 1/sqrt(2))}
\]

or

\[
\text{>> subs(f, 'C1', -1/sqrt(2))}
\]

We get:
\[
y = 2 \tan t
\]
25. Find the solution to the following initial value problem:

\[ y'' = ty' + y + 1, \quad y(0) = 1, \quad y'(0) = 0 \]

**Answer:** We enter into Matlab:

```matlab
>> dsolve('D2y = t*Dy + y + 1', 'y(0) = 1', 'Dy(0) = 1', 't')
>> simple(ans)
```

We enter the `simple` command, because otherwise the answer can obviously be simplified. We get:

\[ y = 2e^{t^2/2} - 1 \]

NSS 4.3

1. Determine whether the following two functions are linearly independent on the interval \((0, 1)\). Also compute the Wronskian \(W[y_1, y_2](x)\).

\[ y_1(x) = e^{-x} \cos 2x, \quad y_2(x) = e^{-x} \sin 2x \]

**Answer:** These two functions are linearly independent.

When \(x = 0\), we have \(y_1(0) = 1\) and \(y_2(0) = 0\). When \(x = \pi/4\), we have \(y_1(\pi/4) = 0\) and \(y_2(\pi/4) = e^{-\pi/4}\). Thus, it is clear that the two functions are not multiples of one another, so they are linearly independent. (Since we are interested in linear independence on the interval \((0, 1)\), we need to choose values of \(x\) from this interval. It is okay to choose \(x = 0\) even though it is not in the interval, because the two functions are continuous. If two continuous functions are multiples of each other on the interval \((0, 1)\), then they would also be multiples of each other on the interval \([0, 1]\).)

The Wronskian of these two functions is

\[
W[y_1, y_2](x) = \begin{vmatrix}
  e^{-x} \cos 2x & e^{-x} \sin 2x \\
  -e^{-x} \cos 2x - 2e^{-x} \sin 2x & -e^{-x} \sin 2x + e^{-x} \cos 2x
\end{vmatrix}
\]

\[
= -e^{-2x} \cos 2x + 2e^{-2x} \cos^2 2x + e^{-2x} \cos 2x \sin 2x + 2e^{-2x} \sin^2 2x \\
= 2e^{-2x} (\cos^2 2x + \sin^2 2x) \\
= 2e^{-2x}
\]

Since \(2e^{-2x}\) is not 0, we have another proof that the functions are linearly independent.
2. Determine whether the following two functions are linearly independent on the interval \((0, 1)\). Also compute the Wronskian \(W[y_1, y_2](x)\).

\[ y_1(x) = e^{3x}, \quad y_2(x) = e^{-4x} \]

**Answer:** These two functions are linearly independent.

When \(x = 0\), we have \(y_1(0) = 1\) and \(y_2(0) = 1\). When \(x = 1\), we have \(y_1(1) = e^3\) and \(y_2(1) = e^{-4}\). Thus, it is clear that the two functions are not multiples of one another, so they are linearly independent. (As in problem 1, it is okay to choose \(x = 0\) and \(x = 1\), since the two functions are continuous.)

The Wronskian of these two functions is

\[
W[y_1, y_2](x) = \begin{vmatrix} e^{3x} & e^{-4x} \\ 3e^{3x} & -4e^{-4x} \end{vmatrix} = -4e^{3x}e^{-4x} - 3e^{3x}e^{-4x} = -4e^{-x} - 3e^{-x} = -7e^{-x}
\]

Since \(-7e^{-x}\) is not 0, we have another proof that the functions are linearly independent.

4. Determine whether the following two functions are linearly independent on the interval \((0, 1)\). Also compute the Wronskian \(W[y_1, y_2](x)\).

\[ y_1(x) = x^2 \cos(\ln x), \quad y_2(x) = x^2 \sin(\ln x) \]

**Answer:** These two functions are linearly independent.

When \(x = 1\), we have \(y_1(1) = 1\) and \(y_2(1) = 0\). When \(x = e^{-\pi/2}\), we have \(y_1(e^{-\pi/2}) = 0\) and \(y_2(e^{-\pi/2}) = -e^{-\pi}\). Thus, it is clear that the two functions are not multiples of one another, so they are linearly independent. (Note: \(e^{-\pi/2}\) was chosen since it was a number in the interval \((0, 1)\) for which it would be easy to compute \(\cos(\ln x)\) and \(\sin(\ln x)\). Also, as in problem 1, it is okay to use \(x = 1\) since the two functions are continuous. It would not work to choose \(x = 0\), because the two functions are not defined at \(x = 0\).)

The Wronskian of these two functions is

\[
W[y_1, y_2](x) = \begin{vmatrix} x^2 \cos(\ln x) & x^2 \sin(\ln x) \\ -x \sin(\ln x) + 2x \cos(\ln x) & x \cos(\ln x) + 2x \sin(\ln x) \end{vmatrix} = x^3 \cos^2(\ln x) + 2x^3 \cos(\ln x) \sin(\ln x) + x^3 \sin^2(\ln x) - 2x^3 \cos(\ln x) \sin(\ln x) = x^3(\cos^2(\ln x) + \sin^2(\ln x)) = x^3
\]

Since \(x^3\) is not 0, we have another proof that the functions are linearly independent.
5. Determine whether the following two functions are linearly independent on the interval $(0, 1)$. Also compute the Wronskian $W[y_1, y_2](x)$.

\[ y_1(x) = \tan^2 x - \sec^2 x, \quad y_2(x) = 3 \]

**Answer:** These two functions are linearly dependent, because $-3(\tan^2 x - \sec^2 x) = 3$, so the two functions are multiples of each other (recall the trig identity $\tan^2 x + 1 = \sec^2 x$).

The Wronskian of these two functions is

\[
W[y_1, y_2](x) = \begin{vmatrix} 
\tan^2 x - \sec^2 x & 3 \\
2 \tan x \sec^2 x - 2 \sec^2 x \tan x & 0 
\end{vmatrix}
= \begin{vmatrix} 
\tan^2 x - \sec^2 x & 3 \\
0 & 0 
\end{vmatrix}
= 0
\]

Since $W[y_1, y_2](x) = 0$, we have another proof that the functions are linearly dependent.

6. Determine whether the following two functions are linearly independent on the interval $(0, 1)$. Also compute the Wronskian $W[y_1, y_2](x)$.

\[ y_1(x) = 0, \quad y_2(x) = e^x \]

**Answer:** These two functions are linearly dependent, because $0(e^x) = 0$, so the two functions are multiples of each other.

The Wronskian of these two functions is

\[
W[y_1, y_2](x) = \begin{vmatrix} 
0 & e^x \\
0 & e^x 
\end{vmatrix}
= 0
\]

Since $W[y_1, y_2](x) = 0$, we have another proof that the functions are linearly dependent.

24. **Linear Dependence of Three Functions.** Three functions $y_1(x), y_2(x),$ and $y_3(x)$ are said to be **linearly independent on an interval** $I$ if, on $I$, at least one of these functions is a linear combination of the remaining two; that is, there exist constants $C_1, C_2, C_3$, not all zero, such that

\[ C_1y_1(x) + C_2y_2(x) + C_3y_3(x) = 0 \]

for all $x$ in $I$. Otherwise, we say they are **linearly dependent**.
(a) Show that if \( y_1 \) and \( y_2 \) are two linearly dependent functions on \( I \), then \( y_1 \) and \( y_2 \), and \( y_3 \) are linearly dependent on \( I \) for any function \( y_3 \).

(b) Show that \( y_1(x) = e^x \), \( y_2(x) = e^{2x} \), and \( y_3(x) = e^{-3x} \) are linearly independent on \(( -\infty, \infty) \).

(c) Show that \( y_1(x) = \cos 2x \), \( y_2(x) = \sin^2 x \), and \( y_3(x) = \cos^2 x \) are linearly dependent on \(( -\infty, \infty) \).

Answer:

(a) If \( y_1 \) and \( y_2 \) are linearly dependent, then they are multiples of each other, so there exists \( c \) such that

\[ cy_1 = y_2 \]

(or there exists \( c \) such that \( cy_2 = y_1 \)— just switch which function is \( y_1 \) and which is \( y_2 \) in this case). Thus:

\[ cy_1 - y_2 = 0 \]

Thus:

\[ cy_1 - y_2 + 0y_3 = 0 \]

Thus, \( C_1 = c \), \( C_2 = -1 \), and \( C_3 = 0 \) gives the linear dependence:

\[ C_1 y_1 + C_2 y_2 + C_3 y_3 = 0 \]

Thus, we have shown that if \( y_1 \) and \( y_2 \) are linearly dependent, then \( y_1 \), \( y_2 \), and \( y_3 \) are linearly dependent for any function \( y_3 \).

(b) We would like to show that \( y_1(x) = e^x \), \( y_2(x) = e^{2x} \), and \( y_3(x) = e^{-3x} \) are linearly independent. So, we need to show that if

\[ C_1 e^x + C_2 e^{2x} + C_3 e^{-3x} = 0 \]

then \( C_1 = 0 \), \( C_2 = 0 \) and \( C_3 = 0 \). We can plug some values in for \( x \) into the above equation. If we plug in \( x = 0 \), \( x = \ln 2 \) and \( x = \ln 3 \), we get the equations:

\[ C_1 + C_2 + C_3 = 0 \]
\[ 2C_1 + 4C_2 + \frac{C_3}{8} = 0 \]
\[ 3C_1 + 9C_2 + \frac{C_3}{27} = 0 \]

If you solve the above equations (using Matlab or by hand) you will get that the only solutions are \( C_1 = 0 \), \( C_2 = 0 \), and \( C_3 = 0 \). Thus, we see that if \( C_1 e^x + C_2 e^{2x} + C_3 e^{-3x} = 0 \), then \( C_1 = 0 \), \( C_2 = 0 \), and \( C_3 = 0 \). Thus, the three functions are linearly independent.

(c) Recall the trig identity \( \cos 2x = \cos^2 x - \sin^2 x \) (if you don’t recall that trig identity, you can use the trig identities \( \cos^2 x = \frac{1}{2} (1 - \cos 2x) \) and \( \cos^2 x + \sin^2 x = 1 \) to derive this trig identity). Thus:

\[ \cos 2x + \sin^2 x - \cos^2 x = 0 \]

Thus, \( C_1 = 1 \), \( C_2 = 1 \), and \( C_3 = -1 \) give the linear dependence. We see that the functions are linearly dependent.
21. First Order Constant Coefficient Equations.

(a) Substituting \( y = e^{rx} \), find the auxiliary equation for the first order linear equation

\[ ay' + by = 0 \]

where \( a, b \) are constants with \( a \neq 0 \).

(b) Use the result of part (a) to find the general solution.

\[ \text{Answer:} \]

(a) We substitute \( y = e^{rx} \), \( y' = re^{rx} \) into the equation, we get

\[ are^{rx} + be^{rx} = 0 \]

Thus:

\[ (ar + b)e^{rx} = 0 \]

The auxiliary equation is \( ar + b \).

A different way to do this (using the methods given in class) is to write the differential equation as

\[ (aD + b)[y] = 0 \]

The, the auxiliary equation is \( ar + b \).

(b) From part (a), we want \( r \) such that \( ar + b = 0 \), which means \( r = -b/a \). Thus, a general solution to the differential equation is \( y = Ae^{-bx/a} \).

54. Consider the differential equation \( y'' - s^2y = 0 \), where \( s \) is a positive constant.

(a) Show that a general solution can be written as \( c_1e^{sx} + c_2e^{-sx} \).

(b) Show that the form \( d_1 \cosh sx + d_2 \sinh sx \) is also a general solution.

(c) Use each of these general solution formats to solve the initial value problem

\[ y'' - s^2y = 0, \quad y(0) = a, \ y'(0) = b. \] Which is more convenient?

\[ \text{Answer:} \]

(a) The auxiliary equation for the differential equation is \( r^2 - s^2 = 0 \). The solutions to this equation are \( r = s \) and \( r = -s \). Thus, the general solution to the equation is

\[ y = c_1e^{sx} + c_2e^{-sx} \]
We can show that the functions $\cosh(sx)$ and $\sinh(sx)$ are also solutions to the equations. If you have not seen the functions $\cosh$ and $\sinh$ previously or do not remember them, they are the hyperbolic cosine and hyperbolic sine and can be defined as follows:

\[
\cosh x = \frac{e^x + e^{-x}}{2} \\
\sinh x = \frac{e^x - e^{-x}}{2}
\]

You can find information about these functions in your calculus textbook or on Wikipedia. One nice fact is that $\frac{d}{dx} \cosh x = \sinh x$ and $\frac{d}{dx} \sinh x = \cosh x$ (you can check these facts with the above definition). Given this, we can see that if $y = \cosh(sx)$, then $y'' - s^2 y = s^2 \cosh(sx) - s^2 \cosh(sx) = 0$ and if $y = \sinh(sx)$, then $y'' - s^2 y = s^2 \sinh(sx) - s^2 \sin(sx) = 0$. Thus, $\cosh(sx)$ and $\sinh(sx)$ are solutions to the differential equation $y'' - s^2 y = 0$. Additionally, they are linearly independent (as they are not multiples of each other). Thus, we can also write the general solution to the differential equation as

$$y = d_1 \cosh sx + d_2 \sinh sx$$

Note, this doesn’t contradict part (a) above. In fact, what this means is that the two solutions must be equivalent. Every solution that you can obtain with the first solution must be obtainable with the second (using some choice of $d_1$ and $d_2$) and every solution that you can obtain with the second solution must be obtainable with the first (using some choice of $c_1$ and $c_2$).

(c) Using the first general solution, we would like to solve the differential equation with initial conditions $y(0) = a$, $y'(0) = b$. Since $y = c_1e^{sx} + c_2e^{-sx}$, we have $y' = c_1se^{sx} - c_2se^{-sx}$. Thus, the initial conditions give us the equations:

\[
a = c_1 + c_2 \\
b = c_1s - c_2s
\]

We can solve these two equations for $c_1$ and $c_2$. We get $c_1 = \frac{as + b}{2s}$ and $c_2 = \frac{as - b}{2s}$. Thus, the solution to the initial value problem is

$$y = \left(\frac{as + b}{2s}\right)e^{sx} + \left(\frac{as - b}{2s}\right)e^{-sx}$$

Using the second general solution, we would like to solve the differential equation with initial conditions $y(0) = a$, $y'(0) = b$. Since $y = d_1 \cosh(sx) + d_2 \sinh(sx)$, we have $y' = d_1s \sinh(sx) + d_2s \cosh(sx)$. Thus, the initial conditions give us the equations:

\[
a = d_1 \\
b = d_2s
\]
Thus, $d_1 = a$ and $d_2 = b/s$. Thus, the solution to the initial value problem is

$$y = a \cosh(sx) + \frac{b}{s} \sinh(sx)$$

The equations involved are certainly easier to solve using \(\cosh\) and \(\sinh\) instead of \(e^{sx}\) and \(e^{-sx}\).

**NSS 4.6**

28. To see the effect of changing the parameter \(b\) in the initial value problem

$$y'' + by' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

solve the problem for \(b = 5, 4, \) and \(2\) and sketch the solutions.

**Answer:** The auxiliary equation is \(r^2 + br + 4 = 0\). When \(b = 5\), the solutions to the auxiliary equation are \(r = -1\) and \(r = -4\). Thus, the general solution to the differential equation (when \(b = 5\)) is

$$y = c_1 e^{-x} + c_2 e^{-4x}$$

The initial conditions \(y(0) = 1\) and \(y'(0) = 0\) give us the equations \(1 = c_1 + c_2\) and \(0 = -c_1 - 4c_2\). Solving these equations, we get \(c_1 = 4/3\) and \(c_2 = -1/3\). Thus, the solution for \(b = 5\) with the given initial conditions is

$$y = \frac{4}{3} e^{-x} - \frac{1}{3} e^{-4x}$$

If we plot this using \texttt{ezplot} on the interval \([0, 10]\) we get the following graph:
When $b = 4$, the solution to the auxiliary equation is $r = -2$. Thus, the general solution to the differential equation (when $b = 4$) is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

The initial conditions $y(0) = 1$ and $y'(0) = 0$ give us the equations $1 = c_1$ and $0 = -2c_1 + c_2$. Solving these equations, we get $c_1 = 1$ and $c_2 = 2$. Thus, the solution for $b = 4$ with the given initial conditions is

$$y = e^{-2x} + 2xe^{-2x}$$

If we plot this using `ezplot` on the interval $[0, 10]$, we get the following graph:

When $b = 2$, the solution to the auxiliary equation is $r = -1 \pm i\sqrt{3}$. Thus, the general solution to the differential equation (when $b = 2$) is

$$y = c_1 e^{-x} \cos(\sqrt{3}x) + c_2 e^{-x} \sin(\sqrt{3}x)$$

The initial conditions $y(0) = 1$ and $y'(0) = 0$ give us the equations $1 = c_1$ and $0 = -c_1 + \sqrt{3}c_2$. Solving these equations, we get $c_1 = 1$ and $c_2 = 1/\sqrt{3}$. Thus, the solution for $b = 2$ with the given initial conditions is

$$y = e^{-x} \cos(\sqrt{3}x) + \frac{1}{\sqrt{3}} e^{-x} \sin(\sqrt{3}x)$$

If we plot this using `ezplot` on the interval $[0, 10]$, we get the following graph:
38. **RLC Series Circuit.** In the study of an electric circuit consisting of a resistor, capacitor, inductor, and an electromotive force, we are led to an initial value problem of the form

\[ L \frac{dI}{dt} + RI + \frac{q}{C} = E(t) \]

\[ q(0) = q_0 \]

\[ I(0) = I_0 \]

where \( L \) is the inductance in henrys, \( R \) is the resistance in ohms, \( C \) is the capacitance in farads, \( E(t) \) is the electromotive force in volts, \( q(t) \) is the charge in coulombs on the capacitor at time \( t \), and \( I = \frac{dq}{dt} \) is the current in amperes. Find the current at time \( t \) if the charge on the capacitor is initially zero, the initial current is zero, \( L = 10 \) henrys, \( R = 20 \) ohms, \( C = (6260)^{-1} \) farads, and \( E(t) = 100 \) volts. [Hint: Differentiate both sides of the differential equation to obtain a homogeneous linear second order equation for \( I(t) \). Then use the equation to determine \( dI/dt \) at \( t = 0 \).]

**Answer:**

First, we plug in the given values:

\[ 10 \frac{dI}{dt} + 20I + 6260q = 100 \]

Since \( I = \frac{dq}{dt} \), it initially looks like we can solve this equation immediately. However, the equation is not homogeneous. Following the hint, we begin by differentiating both sides:

\[ 10 \frac{d^2I}{dt^2} + 20 \frac{dI}{dt} + 6260 \frac{dq}{dt} = 0 \]
Since $I = \frac{dq}{dt}$, this is equivalent to

$$10 \frac{d^2 I}{dt^2} + 20 \frac{dI}{dt} + 6260 I = 0$$

This is a second order linear homogeneous equation with linear coefficients, so we can solve it. The auxiliary equation is $10 r^2 + 20 r + 6260 = 0$. The solutions to this equation are $r = -1 \pm 25i$. Thus, the solutions to the differential equation are

$$I = c_1 e^{-t} \cos(25t) + c_2 e^{-t} \sin(25t)$$

We need initial conditions to solve for $c_1$ and $c_2$. We have that $I(0) = I_0 = 0$. We also have that $q(0) = q_0 = 0$. We need to know $I'(0)$. We can use the original differential equation to solve for $I'(0)$. The original differential equation tells us that

$$\frac{dI}{dt} = 10 - 2I - 626q$$

Thus,

$$\frac{dI}{dt}(0) = 10 - 2I(0) - 626q(0) = 10$$

Since, $I = c_1 e^{-t} \cos(25t) + c_2 e^{-t} \sin(25t)$, we know that

$$\frac{dI}{dt} = -c_1 e^{-t} \cos(25t) - 25c_1 e^{-t} \sin(25t) - c_2 e^{-t} \sin(25t) + 25c_2 e^{-t} \cos(25t)$$

Thus, the initial conditions give us the equations:

$$0 = c_1$$
$$10 = -c_1 + 25c_2$$

Thus, $c_1 = 0$ and $c_2 = 2/5$. Thus, the current at time $t$ is

$$I(t) = \frac{2}{5} e^{-t} \sin(25t)$$

39. **Swinging Door.** The motion of a swinging door with an adjustment screw that controls the amount of friction on the hinges is governed by the initial value problem

$$I \ddot{\theta} + b \dot{\theta} + k \theta = 0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = v_0$$

where $\theta$ is the angle that the door is open, $I$ is the moment of inertia of the door about its hinges, $b > 0$ is a damping constant that varies with the amount of friction on the door, $k > 0$ is the spring constant associated with the swinging door, $\theta_0$ is the initial angle that the door is opened, and $v_0$ is the initial angular velocity imparted to the door. If $I$ and $k$ are fixed, determine for which values of $b$ the door will not continually swing back and forth when closing.

**Answer:**
When solutions to the auxiliary equation are real, the solutions to the differential equation are of the form \(c_1e^{r_1t} + c_2e^{r_2t}\) (or \(c_1e^{r_1t} + c_2te^{r_1t}\) if the auxiliary equation has only one root) with \(r_1 < 0\) and \(r_2 < 0\) \((r_1\) and \(r_2\) are both negative, since \(b > 0\) and \(k > 0\)). Solutions of this form will involve the door slowly closing, and the door will not continually swing back and forth.

When solutions to the auxiliary equation are complex, the solutions to the differential equation are of the form \(c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)\). These solutions do involve the door continually swinging back and forth (because of the \(\cos\) and \(\sin\)).

The auxiliary equation is \(Ir^2 + br + k = 0\). Solutions to this equation are complex when \(b^2 - 4Ik < 0\), and they are real when \(b^2 - 4Ik \geq 0\). Thus, the door will not continually swing back and forth for \(b \geq 2\sqrt{Ik}\).