Matlab 2

34. Find the general solution of the differential equation. Then plot the family of solutions with the indicated initial values over the specified interval. We will use MATLAB notation to indicate the range of initial values. You can use the method of Example 7, but think about using a for loop.

\[ y' + y = \sin t \] on the interval \([0, 4\pi]\) with initial conditions \(y(0) = -10 : 2 : 10\)

**Answer:**
First, we solve the differential equation. It is a linear differential equation, so we use an integrating factor:

\[ \mu(t) = e^{\int dt} = e^t \]

This gives:

\[ \frac{d}{dt} (e^t y) = e^t \sin t \]

Thus:

\[ e^t y = \int e^t \sin t \, dt \]

This integral can be integrated using integration by parts (see your Calculus textbook for more information — the Math 152 syllabus often skips over this sort of integration by parts), but we will just use Matlab (the command is `int('exp(t)*sin(t)','t')`), and we get:

\[ e^t y = -\frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t + C \]

Thus:

\[ y = -\frac{1}{2} \cos t + \frac{1}{2} \sin t + Ce^{-t} \]

If the initial condition is \(y(0) = y_0\), then

\[ y_0 = -\frac{1}{2} \cos(0) + \frac{1}{2} \sin(0) + C e^0 \]

\[ y_0 = -\frac{1}{2} + C \]

\[ C = y_0 + \frac{1}{2} \]

Thus, the solution is

\[ y = -\frac{1}{2} \cos t + \frac{1}{2} \sin t + \left(y_0 + \frac{1}{2}\right) e^{-t} \]
We want to plot these solutions with $y_0 = -10, -8, -6, \ldots, 6, 8, 10$. Here are 3 ways to do this in Matlab. All of these use the plot command instead of the ezplot command. The second two use a for loops. You will want to create these as script M-files, and then run the script M-files. You will also want to close the current figure window, so that it doesn’t plot the resulting graphs on top of your previous figure.

**Example 1:**

```matlab
hold on;
t=linspace(0, 4*pi, 300);
y0 = -10; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 = -8; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 = -6; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 = -4; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 = -2; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 =  0; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 =  2; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 =  4; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 =  6; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 =  8; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
y0 = 10; plot(t, -0.5*cos(t) + 0.5*sin(t) + (y0+0.5).*exp(-t));
grid on
xlabel('t'); ylabel('y');
title('Solutions to $y'' + y = \sin t$');
shg
```

**Example 2:**

```matlab
hold on;
t=linspace(0, 4*pi, 300);
for i=-5:5
    y0=2*i;
    plot(t, -0.5.*cos(t) + 0.5.*sin(t) + (y0+0.5).*exp(-t));
end
grid on
xlabel('t'); ylabel('y');
title('Solutions to $y'' + y = \sin t$');
shg
```
Example 3:

hold on;
\texttt{t=linspace(0, 4*\texttt{pi}, 300);}
\texttt{Y=[ ];}
\texttt{for \textit{i}=-5:5}
\hspace{1em} \texttt{y0=2*\textit{i};}
\hspace{1em} \texttt{Y=[Y; -0.5.*cos(t) + 0.5.*sin(t) + (y0+0.5).*exp(-t)];}
\texttt{end}
\texttt{plot(t, Y);}
\texttt{grid on}
\texttt{xlabel(’t’); ylabel(’y’); title(’Solutions to \texttt{y’} + y = \texttt{sin t}’);
\texttt{shg}

The result of these programs should be a figure that looks like:

For the 3rd example script M-file, the different solutions are different colors.
38. Find the general solution of the differential equation. Then plot the family of solutions with the indicated initial values over the specified interval. We will use MATLAB notation to indicate the range of initial values. You can use the method of Example 7, but think about using a for loop.

\[ y' = y \cos t - 3y \] on the interval \([0, 3]\) with initial conditions \(y(0) = -0.4:0.1:0.4\)

**Answer:**

First, we solve the differential equation. It is a separable differential equation:

\[
\int \frac{dy}{y} = \int (\cos t - 3) \, dt
\]

\[
\ln |y| = \sin t - 3t + C
\]

\[
y = \pm Ce^{\sin t-3t}
\]

\[
y = Ae^{\sin t - 3t}
\]

If the initial condition is \(y(0) = y_0\), then

\[
y_0 = Ae^{\sin(0)-3.0}
\]

\[
y_0 = A
\]

Thus, the solution is

\[
y = y_0e^{\sin t - 3t}
\]

We want to plot these solutions with \(y_0 = -0.4, -0.3, -0.2, \ldots, 0.2, 0.3, 0.4\). Here are 3 ways to do this in Matlab. All of these use the **plot** command instead of the **ezplot** command. The second two use a for loops. You will want to create these as script M-files, and then run the script M-files. You will also want to close the current figure window, so that it doesn’t plot the resulting graphs on top of your previous figure.

**Example 1:**

```matlab
hold on;
t=0:0.01:3;
y0 = -.4; plot(t, y0.*exp(sin(t)-3.*t));
y0 = -.3; plot(t, y0.*exp(sin(t)-3.*t));
y0 = -.2; plot(t, y0.*exp(sin(t)-3.*t));
y0 = -.1; plot(t, y0.*exp(sin(t)-3.*t));
y0 = 0; plot(t, y0.*exp(sin(t)-3.*t));
y0 = .1; plot(t, y0.*exp(sin(t)-3.*t));
y0 = .2; plot(t, y0.*exp(sin(t)-3.*t));
y0 = .3; plot(t, y0.*exp(sin(t)-3.*t));
y0 = .4; plot(t, y0.*exp(sin(t)-3.*t));
grid on
xlabel('t'); ylabel('y');
title(['Solutions to y'' = y\cos(t) - 3y']);
shg
```
Example 2:

```matlab
hold on;
t=0:0.01:3;
for i=-4:4
    y0=i/10;
    plot(t, y0.*exp(sin(t)-3.*t));
end
grid on
xlabel('t'); ylabel('y');
title('Solutions to y'' = ycos(t) - 3y');
shg
```

Example 3:

```matlab
hold on;
t=0:0.01:3;
Y=[];
for i=-4:4
    y0=i/10;
    Y=[Y; y0.*exp(sin(t)-3.*t)];
end
plot(t, Y);
grid on
xlabel('t'); ylabel('y');
title('Solutions to y'' = ycos(t) - 3y');
shg
```
The result of these programs should be a figure that looks like:

![Graph showing solutions to $y' = y\cos(t) - 3y$]

For the 3rd example script M-file, the different solutions are different colors.

**Matlab 10**

1. Use MATLAB and the technique demonstrated in Examples 1 and 2 to verify that $y$ is a solution to the indicated equation.

\[ y = 1 + e^{-t^2/2}, \quad y' + ty = t \]

**Answer:** Enter the following in Matlab:

```
>> syms t
>> y = 1 + exp(-t^2/2)
>> diff(y,t) + t*y - t
>> simple(ans)
```

The result is 0, which tells you that $y$ is a solution to the differential equation.
2. Use MATLAB and the technique demonstrated in Examples 1 and 2 to verify that \( y \) is a solution to the indicated equation.

\[
w = \frac{1}{s - 3}, \quad w' + w^2 = 0
\]

**Answer:** Enter the following in Matlab:

```matlab
>> syms s
>> w = 1/(s-3)
>> diff(w,s) + w^2
```

The result is 0, which tells you that \( w \) is a solution to the differential equation.

6. Determine the independent variable, and use `dsolve` to find the general solution to the indicated equation. Use the `subs` command to replace the integration constant \( C_1 \) with \( C_1 = 2 \). Use `ezplot` to plot the resulting solution.

\[
y' + ty = t
\]

**Answer:** The independent variable is \( t \). We enter the following:

```matlab
>> f = dsolve('Dy+t*y=t', 't')
```

Matlab outputs:

\[
f = 1 + \exp(-t^2/2) * C_1
\]

Thus, the general solution is \( y = 1 + Ce^{-t^2/2} \). Now, enter:

```matlab
>> g = subs(f, 'C1', 2)
>> ezplot(g)
```
7. Determine the independent variable, and use `dsolve` to find the general solution to the indicated equation. Use the `subs` command to replace the integration constant $C_1$ with $C_1 = 2$. Use `ezplot` to plot the resulting solution.

$$y' + y^2 = 0$$

**Answer:** The independent variable does not appear in the differential equation, so we can use whatever variable we want. We will use $t$.

```matlab
>> f = dsolve('Dy+y^2=0', 't')
```

Matlab outputs:

```plaintext
f =

1/(t+C1)
```

Thus, the general solution is $y = \frac{1}{t+C}$. Now, enter:

```matlab
>> g = subs(f, 'C1', 2)
>> ezplot(g)
```
12. Determine the independent variable and use `dsolve` to find the solution to the indicated initial value problem. Use `ezplot` to plot the solution over the indicated time interval.

\[ y' + ty = t, \quad y(0) = -1, \quad [-4, 4] \]

**Answer** The independent variable is \( t \). We enter the following:

```matlab
>> f = dsolve('Dy + t*y = t', 'y(0) = -1', 't')
```

Matlab outputs:

\[ f = 1 - 2*exp(-1/2*t^2) \]

Thus, the solution to the initial value problem is \( y = 1 - 2e^{-t^2/2} \). Now, we enter:

```matlab
>> ezplot(f, [-4, 4])
```
The resulting plot is

![Plot of 1-2 \exp(-1/2 t^2)](image)

13. Determine the independent variable and use dsolve to find the solution to the indicated initial value problem. Use ezplot to plot the solution over the indicated time interval.

\[ y' + y^2 = 0, \quad y(0) = 2, \quad [0, 5] \]

**Answer:** The independent variable does not appear in the differential equation, so we can use whatever variable we want. We will use \( t \).

\[ f = \text{dsolve('Dy + y^2 = 0', 'y(0) = 2', 't')} \]

Matlab outputs:

\[ f = \frac{1}{t + 1/2} \]

Thus, the solution to the initial value problem is \( y = \frac{1}{t + 1/2} \). Now, we enter:

\[ \text{ezplot(f, [0, 5])} \]
The resulting plot is:

```
1/(t+1/2)

0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
0.2 0.4 0.6 0.8 1 1.2 1.4
t
1/(t+1/2)
```

18. Use `dsolve` to obtain the solution to the indicated second order differential equations. Use the `simple` command to find the simplest form of that solution. Use `ezplot` to sketch the solution on the indicated time interval.

\[ y'' + 4y = 3\cos(2.1t), \quad y(0) = 0, \quad y'(0) = 0, \quad [0, 64\pi] \]

**Answer:** We enter

```
>> f = dsolve('D2y+4*y=3*cos(2.1*t)', 'y(0)=0', 'Dy(0) = 0', 't')
```

Matlab returns

```
f =
300/41*cos(2*t)-300/41*cos(21/10*t)
```

If you enter `simple(f)`, it returns the same thing, because there is not a way to simplify the expression. Next, we enter

```
>> ezplot(f, [0, 64*pi])
```
26. Suppose we start with a population of 100 individuals at time $t = 0$, and that the population is correctly modelled by the logistic equation. Suppose that at time $t = 2$ there are 200 individuals in the population, and that the population approaches the steady state at a population of 1000. Plot the population over the interval $[0, 20]$. What is the population at time $t = 10$?

**Answer:** Recall that the logistic equation is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Since, the population approaches the steady state at a population of 1000, we know that $K = 1000$. Also, we know that $P(0) = 100$. Thus, we have the following initial value problem, which we can enter into Matlab:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{1000}\right), \quad P(0) = 100$$

We enter into Matlab the following:

```matlab
>> f = dsolve('DP = r*P*(1 - P/1000)', 'P(0)=100', 't')
```

Matlab outputs:
f = 

\frac{1000}{1 + 9e^{-rt}}

Thus, we know that solutions are of the form \( P = \frac{1000}{1 + 9e^{-rt}} \). We need to solve for \( r \) using the fact that \( P(2) = 200 \). We can use the \texttt{subs} command to substitute \( t = 2 \) into the solution, and then we can use the \texttt{solve} command to find the value of \( r \) for \( P(2) = 200 \):

\[
\begin{align*}
\text{>> } g &= \text{subs}(f, 't', 2) \\
\text{>> } h &= g - 200 \\
\text{>> } a &= \text{solve}(h, 'r')
\end{align*}
\]

This finds the value of \( r \) for which \( P(2) - 200 = 0 \). Matlab outputs

\[
\text{ans = -1/2*log(4/9)}
\]

Thus, \( r = \frac{\ln(4/9)}{2} \approx 0.4055 \). We can substitute this value of \( r \) into the solution (recall that \( f \) was the solution and \( a \) is the value of \( r \)):

\[
\begin{align*}
\text{>> } P &= \text{subs}(f, 'r', a)
\end{align*}
\]

Matlab outputs:

\[
P = \frac{1000}{1 + 9e^{(1/2*\log(4/9))*t}}
\]

This is the equation for the population. We can plot it:

\[
\begin{align*}
\text{>> } \text{ezplot}(P, [0, 20])
\end{align*}
\]
The resulting plot is

And, we can determine the population at \( t = 10 \):

\[
\text{ans} = 864.9967
\]

Thus, the population at time \( t = 10 \) is approximately 864.9967. (Since populations are integers, 864 and 865 would both be acceptable answers.)

27. Suppose we have a population that is correctly modelled by the logistic equation, and the experimental measurements show that \( p(0) = 50 \), \( p(1) = 150 \), and \( p(2) = 250 \). Use the Symbolic Toolbox to derive the formula for the population as a function of time. Plot the population over the interval \([0, 5] \). What is the limiting value of the population? What is \( p(3) \)?

**Answer:** Recall that the logistic equation is

\[
\frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right)
\]
We know that \( P(0) = 50 \). Thus, we have the following initial value problem, which we can enter into Matlab:

\[
\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right), \quad P(0) = 50
\]

We enter into Matlab the following:

```matlab
>> f = dsolve('DP = r*P*(1 - P/K)', 'P(0)=50', 't')
```

Matlab outputs:

\[
f = K/(1+1/50*exp(-r*t)*(K-50))
\]

Thus, we know that the solution to the initial value problem is

\[
P = \frac{K}{1 + (1/50)(K - 50)e^{-rt}}
\]

We need to use the facts \( P(1) = 150 \) and \( P(2) = 250 \) to solve for \( r \) and \( K \). We can use the \texttt{subs} and \texttt{solve} commands to get Matlab to solve for \( r \) and \( K \):

```matlab
>> p1 = subs(f, 't', 1)
>> pop1 = p1 - 150
>> p2 = subs(f, 't', 2)
>> pop2 = p2 - 250
>> sol = solve(pop1, pop2, 'r', 'K')
```

Then, the commands \texttt{sol.r} and \texttt{sol.K} will give you the solutions. We get that \( r = \ln 5 \) and \( K = 300 \). We substitute these into the original solution (recall that \( f \) is the solution):

```matlab
>> P = subs(f, 'r', sol.r)
>> P = subs(P, 'K', sol.K)
>> P = simple(P)
```

We get that

\[
P = \frac{300}{1 + 5^{1-t}}
\]

We can plot this with \texttt{ezplot(P, [0, 5])}.

15
The resulting plot is

![Graph showing population growth](image)

The limiting population is $K = 300$. We can find $P(3)$ using the `subs` command:

```matlab
>> subs(P, 't', 3)
```

We get that $P(3) \approx 288.4615$. (Since populations are integers, the answer is 288.)

**NSS 3.2**

10. Use a sketch of the phase line (see Project D, Chapter 1) to argue that any solution to the mixing model

$$\frac{dx}{dt} = a - bx; \quad a, b > 0$$

approaches the equilibrium solution $x(t) \equiv a/b$ as $t$ approaches $+\infty$; that is, $a/b$ is a sink.

**Answer:**

First, we find the equilibrium solutions: $a - bx = 0$ implies that $x = a/b$. Thus, there is one equilibrium point $x = a/b$.

The graph of $f(x) = a - bx$ is a line with slope $-b$. Since the slope is negative (since $b$ is positive), we know that $f(x)$ is negative for large values of $x$ and positive for small values of $x$. Thus, for $x > a/b$, $a - bx$ is negative, and for $x < a/b$, $a - bx$ is positive. This gives us the following phase line:
We can see from the phase line that \( x = a/b \) is a sink, and that every solution to the differential equation will approach the equilibrium solution \( x = a/b \).

18. A population model used in actuarial predictions is based on the **Gompertz equation**

\[
\frac{dP}{dt} = P(a - b \ln P)
\]

where \( a \) and \( b \) are constants.

(a) Solve the Gompertz equation for \( P(t) \).

(b) If \( P(0) = P_0 > 0 \), give a formula for \( P(t) \) in terms of \( a, b, P_0 \), and \( t \).

(c) Describe the behavior of \( P(t) \) as \( t \to +\infty \). [Hint: Consider the cases for \( b > 0 \) and \( b < 0 \).]

**Answer:**

(a) This is a separable differential equation:

\[
\int \frac{dP}{P(a - b \ln P)} = \int dt
\]

When we separated, we removed the solutions \( P = 0 \) and \( P = e^{a/b} \). We will need to remember to include these solutions in our final answer. (There is some question as to whether \( P = 0 \) is a solution or not, because of the \( \ln P \) in the differential equation. Since \( \lim_{P\to0} P(a - b \ln P) = 0 \), there is argument that it is a solution. Don’t worry about this too much. If it comes up on a test question, I would accept either answer.)

The left side can be integrated with the \( u \) substitution \( u = \ln P \), \( du = dP/P \), or you can just use Matlab with the command `int('1/(P*(a-b*log(P)))','P')`. After integrating, we get:

\[
-\frac{1}{b} \ln |a - b \ln P| = t + C_1
\]
This becomes:

\[ a - b \ln P = \pm e^{C_2}e^{-bt} \]

We can let \( A = \pm e^{C_2} \). This reintroduces the constant solution \( P = e^{a/b} \). Solving for \( P \), we get

\[
\ln(P) = \frac{a}{b} + Ae^{-bt} \\
P = e^{(a/b)}e^{Ae^{-bt}}
\]

Thus, the solutions to the differential equation are

\[
P = e^{a/b}e^{Ae^{-bt}}, \quad P = 0
\]

(b) We can plug \( t = 0 \) and \( P = P_0 \) into the above solution:

\[
P_0 = e^{a/b}e^{Ae^0}
\]

Then,

\[
P_0 = e^{a/b}e^A
\]

Solving for \( A \), we get

\[
A = \ln(P_0e^{-a/b})
\]

Thus, we have

\[
A = \ln(P_0) + \ln(e^{-a/b}) = \ln(P_0) - \frac{a}{b}
\]

Thus, the solution is

\[
P = e^{a/b}e^{(\ln(P_0) - (a/b))e^{-bt}}
\]

This answer is equivalent (after applying a bunch of logarithm rules) to the answer Matlab gives. The Matlab command to get the answer is `dsolve('DP = P*(a-b*log(P))','P(0)=P0','t')`.

(c) The only \( t \) in the expression is in the \( e^{-bt} \). If \( b > 0 \), then as \( t \to \infty \), we have \( e^{-bt} \to 0 \). Thus, for \( b > 0 \)

\[
\lim_{t \to \infty} e^{a/b}e^{(\ln(P_0) - (a/b))e^{-bt}} = e^{a/b}
\]

If \( b < 0 \), then as \( t \to \infty \), we have \( e^{-bt} \to \infty \). In this case, the limit depends on whether \( \ln(P_0) - (a/b) \) is positive or negative. If it is positive, then \( P(t) \to \infty \).

If it is negative, then \( P(t) \to 0 \). Whether \( \ln(P_0) - (a/b) \) is positive or negative depends on whether \( P_0 > e^{a/b} \) or \( P_0 < e^{a/b} \).

Here are the cases:
### Case Analysis

<table>
<thead>
<tr>
<th>Case</th>
<th>What happens as $t \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b &gt; 0$</td>
<td>$P(t) \to e^{a/b}$</td>
</tr>
<tr>
<td>$b &lt; 0, P_0 &gt; e^{a/b}$</td>
<td>$P(t) \to \infty$</td>
</tr>
<tr>
<td>$b &lt; 0, P_0 &lt; e^{a/b}$</td>
<td>$P(t) \to 0$</td>
</tr>
<tr>
<td>$b &lt; 0, P_0 = e^{a/b}$</td>
<td>$P(t) \to e^{a/b}$</td>
</tr>
</tbody>
</table>

By the way, we assumed while solving the differential equation that $b \neq 0$ (notice, in particular, the integration step). This is why it doesn’t make any sense to consider the longterm behavior when $b = 0$. Also, if $b = 0$, then the differential equation is equivalent to the Malthusian model of population growth, which we have already analyzed.

It is also possible to analyze the behavior of $P$ as $t \to \infty$ by drawing the phase lines. Here are the phase lines for the two cases ($b < 0$ and $b > 0$). We only draw the phase line for $P \geq 0$, since the differential equation only makes sense for $P \geq 0$ (due to the $\ln P$).

**By the way, you may be wondering how the units can work out on this differential equation.** It is hard to see how to get the units to work out with the way the equation is written. But, if we change some of the constants, the given differential equation is equivalent to

$$ \frac{dP}{dt} = bP \ln \left( \frac{K}{P} \right) $$

In terms of our original constants, $b$ is the same and $K = e^{ab}$. The quantity $K$ has the same units as $P$ (number of people, fish, etc.), and the quantity $b$ has units $1/time$. 

19
26. To see how sensitive the technique of carbon dating of Problem 25 is,

(a) Redo Problem 25 assuming the half-life of carbon-14 is 5550 yr.
(b) Redo Problem 25 assuming the 3% of the original mass remains.
(c) If each of the figures in parts (a) and (b) represent a 1% error in measuring the
two parameters of half-life and percent of mass remaining, to which parameter
is the model more sensitive?

Answer:

In Problem 25, you should get that
\[ t = -\frac{\ln(0.02)}{\ln 2} \frac{5600}{\ln 2} \approx 31,606 \text{ years.} \]
Note that the 5600 came from the half-life of Carbon-14 and the .02 came from the 2%. Thus, to
answer parts (a) and (b), we just need to modify accordingly.

(a) \[ t = -\frac{\ln(0.02)}{\ln 2} \frac{5550}{\ln 2} \approx 31,323 \text{ years} \]
(b) \[ t = -\frac{\ln(0.03)}{\ln 2} \frac{5600}{\ln 2} \approx 28,330 \text{ years} \]
(c) The model is more sensitive to the percent of mass remaining (since (b) was
farther from 31,606 than (a)).

NSS 3.4

16. Find the equation for the angular velocity \( \omega \) in Problem 15, assuming that the retarding torque is proportional to \( \sqrt{\omega} \).

Answer: By Newton’s second law for rotational motion, we have the differential equation
\[ I \frac{d\omega}{dt} = T_1 + T_2 \]
where \( I \) is the moment of inertia, \( \omega \) is the angular velocity, \( T_1 \) is the torque from
the motor, and \( T_2 \) is the retarding torque. We know that \( \omega(0) = \omega_0 \), \( T_1 = T \), and
\( T_2 = k\sqrt{\omega} \) where \( k \) is the constant of proportionality. Thus, we have the following
initial value problem:
\[ I \frac{d\omega}{dt} = T + k\sqrt{\omega}, \quad \omega(0) = \omega_0 \]
where \( I, T, k, \) and \( \omega_0 \) are constants. This is a separable differential equation:
\[ \int \frac{I}{T + k\sqrt{\omega}} d\omega = \int dt \]
The left side of the equation can be integrated using the substitution \( u = \sqrt{\omega} \) followed
by long division. We will just use Matlab. We get
\[ -\frac{IT}{k^2} \left( \ln(\frac{k^2}{2} - T^2) + \ln(T + k\sqrt{\omega}) - \ln(-T + k\sqrt{\omega}) \right) + \frac{2I\sqrt{\omega}}{k} = t + C \]
We can solve for $C$ using the initial condition $\omega(0) = \omega_0$:

\[
C = \frac{-IT}{k^2} \left( \ln\left(\frac{k^2 \omega_0 - T^2}{T + k\sqrt{\omega_0}}\right) - \ln\left(-T + k\sqrt{\omega_0}\right) \right) + \frac{2I\sqrt{\omega_0}}{k}
\]

18. When an object slides on a surface, it encounters a resistance force called friction. This force has a magnitude of $\mu N$, where $\mu$ is the coefficient of kinetic friction and $N$ is the magnitude of the normal force that the surface applies to the object. Suppose an object of mass 30 kg is released from the top of an inclined plane that is inclined 30° to the horizontal. Assuming the gravitational force is constant, air resistance is negligible, and the coefficient of kinetic friction $\mu = 0.2$. Determine the equation of motion for the object as it slides down the plane. If the top surface of the plane is 5 m long, what is the velocity of the object when it reaches the bottom?

Answer:

If you have not taken an introductory physics course (or if it has been a while since you did), you may find the statement of this problem confusing.

The force due to gravity is a vector with magnitude $(mg)$ and downwards direction. The component of this force in the direction of the surface is $mg \sin(30^\circ)$. This is one of the forces acting on the object.

The component of the gravitational force perpendicular to the surface is the force that the object applies to the surface (this has magnitude $mg \cos(30^\circ)$). The surface applies an equal and opposite force to the object — magnitude $mg \cos(30^\circ)$ and direction perpendicular to the surface. This force is called the normal force. In the statement of the problem, the magnitude of the normal force is denoted by $N$. Thus, $N = mg \cos(30^\circ)$.

The other force acting on the object is friction. It has magnitude $\mu N$ and points upwards parallel to the surface.

Thus, we have two forces: $F_1 = mg \sin(30^\circ)$ is the force due to gravity, and $F_2 = \mu mg \cos(30^\circ)$ is the force due to friction. Applying Newton’s second law, we have the differential equation

\[
m \frac{dv}{dt} = mg \sin(30^\circ) - \mu mg \cos(30^\circ)
\]

Setting $m = 30$ kg, $g = 9.81$ m/s², and $\mu = 0.2$, have

\[
(30) \frac{dv}{dt} = \frac{(30)(9.81)}{2} - \frac{(0.2)(30)(9.81)\sqrt{3}}{2}
\]
Thus,
\[ \frac{dv}{dt} = 3.2059 \]

This problem doesn’t really involve solving a differential equation. We just have to integrate:
\[ v = \int 3.2059 \, dt = 3.2059 \, t + C_1 \]

We know that the initial velocity is 0, so \( C_1 = 0 \):
\[ v = 3.2059 \, t \]

Integrating again, to find position:
\[ x = \int 3.2059 \, t \, dt = 1.6029 \, t^2 + C_2 \]

We should set our coordinate system so that the initial position is 0. Thus, \( C_2 = 0 \).

The equation for the object’s motion as it slides down the plane is:

\[
\begin{align*}
  x(t) &= 1.6029 \, t^2 \\
\end{align*}
\]

We want to find the velocity of the object when it reaches the bottom. To do this, we first need to find the time when the object hits the bottom, which is the time when the object has travelled 5 meters:
\[ 1.6029 \, t^2 = 5 \quad \Rightarrow \quad t = 1.7662 \text{ seconds} \]

Using, the equation for the velocity of the object \( (v = 3.2059 \, t) \), we get that the velocity when the object hits the bottom is \( 5.662 \text{ m/s} \).

**NSS 3.5**

2. An RC circuit with a 1-Ω resistor and a 0.000001-F capacitor is driven by a voltage \( E(t) = \sin 100t \) V. If the initial capacitor voltage is zero, determine the subsequent resistor and capacitor voltages and the current.

**Answer:** The voltage drop across the resistor is \( RI = R \frac{dq}{dt} \). Since \( R = 1-\Omega \), the voltage drop across the resistor is \( \frac{dq}{dt} \). The voltage drop across the capacitor is \( \frac{q}{C} \). Since \( C = .000001 \text{ F} \), the voltage drop across the capacitor is \( \frac{q}{.000001} = 100000 \, q \).

Thus, the total voltage drop is \( \frac{dq}{dt} + 100000 \, q \). By Kirchoff’s voltage law, this must equal \( E(t) = \sin 100t \).

Thus, we get the differential equation
\[ \frac{dq}{dt} + 100000 \, q = \sin(100t) \]
This equation is linear.
\[ \mu(t) = e^{\int_{100000}^{t} \, dt} = e^{t-100000} \]

Thus,
\[ e^{100000t} q = \int e^{100000t} \sin(100t) \, dt \]

The integral on the right-hand side is integrable using integration by parts (see a Calculus textbook — this sort of integration by parts is sometimes skipped in Math 152), but we will just use Matlab (the command is `int('exp(100000*t)*sin(100*t)', 't')`). Use the command `simple(ans)` to make the result of the integration simpler.

We get
\[ e^{100000t} q = -\frac{1}{100,000,100} e^{100000t} (\cos(100t) - 1000 \sin(100t)) + C_1 \]

(The constant is \( C_1 \) just to distinguish it from the capacitance.) Thus:
\[ q = -\frac{\cos(100t) + 1000 \sin(100t)}{100,000,100} + C_1 e^{-100000t} \]

Since the initial capacitor voltage is 0, we know that \( q(0) = 0 \). We can use this initial condition to solve for \( C_1 \):
\[ 0 = \frac{-1}{100,000,100} + C_1 \]

Thus:
\[ C_1 = \frac{1}{100,000,100} \]

Thus, the capacitor charge is
\[ q(t) = -\frac{\cos(100t) + 1000 \sin(100t) + e^{-100000t}}{100,000,100} \]

We are asked to find the resistor voltage \( E_R = RI = R \frac{dq}{dt} \), the capacitor voltage \( E_C = \frac{q}{C} \), and the current \( I = \frac{dq}{dt} \) (recall that \( R = 1 \) and \( C = .000001 \)). We need to compute \( \frac{dq}{dt} \):
\[ \frac{dq}{dt} = \frac{100 \sin(100t) + 100000 \cos(100t) - 100000 e^{-100000t}}{100,000,100} \]
\[ = \frac{\sin(100t) + 1000 \cos(100t) - 1000 e^{-100000t}}{1,000,001} \]

Thus, the answers are:
\( E_R = \frac{dq}{dt} = \frac{\sin(100t) + 1000 \cos(100t) - 1000e^{-100000t}}{1,000,001} \)

\( E_C = 10000q = \frac{-1000 \cos(100t) + 1,000,000 \sin(100t) + 1000e^{-100000t}}{1,000,001} \)

\( I = \frac{dq}{dt} = \frac{\sin(100t) + 1000 \cos(100t) - 1000e^{-100000t}}{1,000,001} \)