NSS 8.2

32. Determine the Taylor series for \( f(x) \) about the point \( x_0 \).

\[ f(x) = \ln(1 + x), \quad x_0 = 0 \]

**Answer:** We can find the Taylor series for \( \ln(1 + x) \) by finding the Taylor series for \( \frac{1}{1 + x} \) and then integrating. The Taylor series for \( \frac{1}{1 + x} \) is:

\[ \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \]

This Taylor series is obtained using the formula for geometric series:

\[ a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots = \frac{a}{1 - r} \]

Now, we can integrate the Taylor series:

\[ \int \frac{dx}{1 + x} = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \ldots) \, dx \]

We get:

\[ \ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \ldots + C \]

We can solve for \( C \) by plugging in \( x = 0 \). When \( x = 0 \), we have \( \ln(1) = C \); thus, \( C = 0 \). Thus, we have the following Taylor series for \( f(x) \) about the point \( x_0 = 0 \):

\[ \ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \ldots \]

We can write this using summation notation:

\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \]
12. In the following problem, (a) verify that the functions $y_1$ and $y_2$ are linearly independent solutions of the given differential equation, (b) find a general solution to the given differential equation, and (c) find the solution that satisfies the given initial condition.

$$y'' - y = 0; \quad y_1(x) = \cosh x, \quad y_2(x) = \sinh x; \quad y(0) = 1, \quad y'(0) = -1$$

**Answer:**

(a) First, we can check that $y_1$ and $y_2$ are solutions to the differential equation.

Recall that:

$$\frac{d}{dx} (\cosh x) = \sinh x$$
$$\frac{d}{dx} (\sinh x) = \cosh x$$

We differentiate $y_1$ and plug into the differential equation:

$$y'_1 = \sinh x$$
$$y''_1 = \cosh x$$

Plugging into the differential equation, we get:

$$\cosh x - \cosh x = 0$$

Thus, $y_1 = \cosh x$ is a solution to the differential equation.

We differentiate $y_2$ and plug into the differential equation:

$$y'_2 = \cosh x$$
$$y''_2 = \sinh x$$

Plugging into the differential equation, we get:

$$\sinh x - \sinh x = 0$$

Thus, $y_2 = \sinh x$ is a solution to the differential equation.

Now, we need to check that $y_1$ and $y_2$ are linearly independent. They are linearly independent as long as they are not multiples of each other. Recall that:

$$\cosh x = \frac{1}{2} e^x + \frac{1}{2} e^{-x}$$
$$\sinh x = \frac{1}{2} e^x - \frac{1}{2} e^{-x}$$

Thus, $\cosh(0) = 1$ and $\sinh(0) = 0$. Also, $\cosh(\ln 2) = \frac{5}{4}$ and $\sinh(\ln 2) = \frac{3}{4}$.

Thus, they are not multiples of each other.

(b) Thus, a general solution to the differential equation is $y = c_1 \cosh x + c_2 \sinh x$
If \( y = c_1 \cosh x + c_2 \sinh x \), we can plug in \( x = 0 \) and use the initial condition \( y(0) = 1 \). We get

\[
1 = c_1 \cosh(0) + c_2 \sinh(0)
\]

Since \( \sinh(0) = 0 \) and \( \cosh(0) = 1 \) (see part (a)), we have

\[
1 = c_1
\]

Also, \( y' = c_1 \sinh x + c_2 \cosh x \). We can plug in \( x = 0 \) and use the initial condition \( y'(0) = -1 \):

\[
-1 = c_1 \sinh(0) + c_2 \cosh(0)
\]

Thus, we get \(-1 = c_2\). Thus, the solution is

\[
y = \cosh x - \sinh x
\]

NSS 4.5

54. Consider the differential equation \( y'' - s^2 y = 0 \), where \( s \) is a positive constant.

(a) Show that a general solution can be written as \( c_1 e^{sx} + c_2 e^{-sx} \).

(b) Show that the form \( d_1 \cosh sx + d_2 \sinh sx \) is also a general solution.

(c) Use each of these general solution formats to solve the initial value problem \( y'' - s^2 y = 0, \ y(0) = a, \ y'(0) = b \). Which is more convenient?

Answer:

(a) The auxiliary equation for the differential equation is \( r^2 - s^2 = 0 \). The solutions to this equation are \( r = s \) and \( r = -s \). Thus, the general solution to the equation is

\[
y = c_1 e^{sx} + c_2 e^{-sx}
\]

(b) We can show that the functions \( \cosh(sx) \) and \( \sinh(sx) \) are also solutions to the equations. If you have not seen the functions \( \cosh \) and \( \sinh \) previously or do not remember them, they are the hyperbolic cosine and hyperbolic sine and can be defined as follows:

\[
\begin{align*}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2}
\end{align*}
\]

You can find information about these functions in your calculus textbook or on Wikipedia. One nice fact is that \( \frac{d}{dx} \cosh x = \sinh x \) and \( \frac{d}{dx} \sinh x = \cosh x \) (you can check these facts with the above definition). Given this, we can see that if \( y = \cosh(sx) \), then \( y'' - s^2 y = s^2 \cosh(sx) - s^2 \cosh(sx) = 0 \) and if \( y = \sinh(sx) \), then \( y'' - s^2 y = s^2 \sinh(sx) - s^2 \sin(sx) = 0 \). Thus, \( \cosh(sx) \) and \( \sinh(sx) \) are solutions to the differential equation \( y'' - s^2 y = 0 \). Additionally, they are linearly independent (as they are not multiples of each other). Thus, we can also write the general solution to the differential equation as

\[
y = c_1 \cosh x + c_2 \sinh x
\]
\[ y = d_1 \cosh sx + d_2 \sinh sx \]

Note, this doesn’t contradict part (a) above. In fact, what this means is that the two solutions must be equivalent. Every solution that you can obtain with the first solution must be obtainable with the second (using some choice of \(d_1\) and \(d_2\)) and every solution that you can obtain with the second solution must be obtainable with the first (using some choice of \(c_1\) and \(c_2\)).

(c) Using the first general solution, we would like to solve the differential equation with initial conditions \(y(0) = a\), \(y'(0) = b\). Since \(y = c_1 e^{sx} + c_2 e^{-sx}\), we have \(y' = c_1 s e^{sx} - c_2 s e^{-sx}\). Thus, the initial conditions give us the equations:

\[
\begin{align*}
  a &= c_1 + c_2 \\
  b &= c_1 s - c_2 s
\end{align*}
\]

We can solve these two equations for \(c_1\) and \(c_2\). We get \(c_1 = \frac{a s + b}{2s}\) and \(c_2 = \frac{a s - b}{2s}\). Thus, the solution to the initial value problem is

\[ y = \left( \frac{a s + b}{2s} \right) e^{sx} + \left( \frac{a s - b}{2s} \right) e^{-sx} \]

Using the second general solution, we would like to solve the differential equation with initial conditions \(y(0) = a\), \(y'(0) = b\). Since \(y = d_1 \cosh(sx) + d_2 \sinh(sx)\), we have \(y' = d_1 s \sinh(sx) + d_2 s \cosh(sx)\). Thus, the initial conditions give us the equations:

\[
\begin{align*}
  a &= d_1 \\
  b &= d_2 s
\end{align*}
\]

Thus, \(d_1 = a\) and \(d_2 = b/s\). Thus, the solution to the initial value problem is

\[ y = a \cosh(sx) + \frac{b}{s} \sinh(sx) \]

The equations involved are certainly easier to solve using \(\cosh\) and \(\sinh\) instead of \(e^{sx}\) and \(e^{-sx}\).
Additional Problems

1. For each of the following Matlab programs, determine how many times the program will print “Hello World”?

   (a) for i=1:5
       disp('Hello World')
   end

   (b) for i=-9:25
       disp('Hello World')
   end

   (c) for i=1:10
       for j=1:3
           disp('Hello World')
       end
   end

Answer:

   (a) This program will print “Hello World” 5 times

   (b) Since this for loop starts with i = -9 and ends with i = 25, it will print “Hello World” 35 times

   (c) This program will print “Hello World” 30 times

2. For each of the following Matlab programs, determine what number the program will output and why.

   (a) a=3;
       for i=1:10
           a=a+2;
       end;
   a

   (b) a=1;
       for i=1:6
           a=a*i;
       end;
   a

   (c) a=0;
       for i=1:5
           for j=3:6
               for k=0:9
                   a=a+1;
               end;
           end;
       end;
   a

   a
Answer:

(a) This program starts with $a = 3$. Each time through the loop, the program adds 2 to $a$. Since the loop is repeated 10 times, the output is $3 + 10 \cdot 2$ which equals 23.

(b) This program starts with $a = 1$. Each time through the loop, the program multiplies $a$ by $i$. Since the loop is repeated 6 times, the output is $6!$ which equals 720.

(c) This program starts with $a = 0$. Then, it adds 1 to $a$ 200 times. Thus, the output is 200.

3. Consider the recursively defined sequence with $a_0 = 2$ and $a_{n+1} = a_n + 3$. Write a program in Matlab to compute $a_{10}$.

Answer: Here is the program:

```matlab
a=2;
for i=1:10
    a=a+3;
end;
a
```

From this program, we get that $a_{10} = 32$.

4. Consider the recursively defined sequence with $a_0 = 1$, $a_1 = -2$, and $a_{n+1} = a_n + 2a_{n-1}$. Write a program in Matlab to compute $a_{15}$.

Answer: Here is the program:

```matlab
a=1;
b=-2;
for i=1:14
    c=b+2*a;
    a=b;
    b=c;
end;
c
```

From this program, we get that $a_{15} = -10924$. 
5. Consider the recursively defined sequence with \(a_0 = 0.5\) and \(a_{n+1} = a_n^2 - a_n\). Find \(a_{100}\) to 6 decimal places.

**Answer:** Here is the program:

```matlab
format long
da=0.5;
for i=1:100
    a=a.^2-a;
end;
a
```

From this program, we get that \(a_{100} = 0.070096\)

6. Consider the recursively defined sequence \(a_0 = 1, a_1 = -1, a_2 = 0\) and \(a_{n+1} = a_n - 2a_{n-1} + 3a_{n-2}\). Write a program in Matlab to compute \(a_{25}\).

**Answer:** Here is the program:

```matlab
a=1;
b=-1;
c=0;
for i=1:23
d=c-2*b+3*a;
a=b;
b=c;
c=d;
end;
d
```

From this program, we get that \(a_{100} = 10232\)