Math 308, Spring 2007
Section 507, Test 3

Name: Solutions

Directions:

1. You may use MATLAB during the test. This is the only program you may have open on the computer. You may not use any other computer programs, and you may not use a calculator.

2. For the first four problems, do not use MATLAB to solve the differential equations; in particular, you may not use the dsolve command (except to check your work). You may use MATLAB to compute integrals, solve equations, perform arithmetic, and check your answers. You must show your work to receive any credit.

3. For the last two problems, you may use any MATLAB commands.

4. For all of the problems, show your work, and clearly indicate your final answer.

For instructor use only:

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
For the first four problems, do not use MATLAB to solve the differential equations; in particular, you may not use the `dsolve` command (except to check your work). You may use MATLAB to compute integrals, solve equations, perform arithmetic, and check your answers. You must show your work to receive any credit.

1. [16 pts.] Solve the following initial value problem:

\[ y'' - 4y' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 4 \]

** auxiliary eq: ** \[ r^2 - 4r + 4 = 0 \]
\[ (r - 2)^2 = 0 \]
\[ r = 2 \]

** General solution ** \[ y = c_1 e^{2x} + c_2 xe^{2x} \]
\[ y' = 2c_1 e^{2x} + 2c_2 xe^{2x} + c_2 e^{2x} \]

** initial condition ** \[ y(0) = 3, \: y'(0) = 4 \]
\[ y(0) = 3 \Rightarrow 3 = c_1 \]
\[ y'(0) = 4 \Rightarrow 4 = 2c_1 + c_2 \]
\[ \begin{align*}
  &c_1 = 3 \\
  &c_2 = -2
\end{align*} \]

** Solution ** \[ y = 3e^{2x} - 2xe^{2x} \]
2. [16 pts.] Solve the following differential equation:

\[ x^2 y'' + 3xy' - 8y = 0 \]

Cauchy–Euler

Substitution

\[ x = e^t \]

\[ x \frac{dy}{dx} = \frac{dy}{dt} \]

\[ x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \]

\[ \frac{d^2y}{dt^2} - \frac{dy}{dt} + 3\frac{dy}{dt} - 8y = 0 \]

Auxiliary eq:

\[ r^2 + 2r - 8 = 0 \]

\[ (r + 4)(r - 2) = 0 \]

\[ r = -4, 2 \]

\[ y = c_1 e^{-4t} + c_2 e^{2t} \]

\[ t = \ln(x) \]

\[ y = c_1 e^{-4\ln x} + c_2 e^{2\ln x} \]

\[ y = \frac{c_1}{x^4} + c_2 x^2 \]
3. [20 pts.] Consider the following differential equation:

\[ xy'' - (2x + 2)y' + 4y = 0, \quad x > 0 \]

One solution to this differential equation is \( f(x) = e^{2x} \). Find all solutions to the differential equation.

Reduction of order:

\[ y = ve^{2x} \]
\[ y' = 2ve^{2x} + ve^{2x} \]
\[ y'' = 4ve^{2x} + 4ve^{2x} + ve^{2x} \]

\( x(4ve^{2x} + 4ve^{2x} + ve^{2x}) - (2x + 2)(2ve^{2x} + ve^{2x}) + 4ve^{2x} = 0 \)

\( \nu \) terms:

\[ x(4ve^{2x} - (2x+2)2ve^{2x} + 4ve^{2x}) = (4x - 4x - 4 + 4)ve^{2x} = 0 \]

\( \nu' \) terms:

\[ x(4ve^{2x} - (2x+2)(ve^{2x})) = (4x - 2x - 2)ve^{2x} = (2x - 2)ve^{2x} \]

\( \nu'' \) terms:

\[ xve^{2x} \]

\( \nu''e^{2x} + (2x - 2)\nu'e^{2x} = 0 \)

\[ u = \nu' \quad u' = \nu'' \]

\[ xu' + (2x - 2)u = 0 \]

\[ x \frac{du}{dx} = (2 - 2x)u \]

\[ \int \frac{du}{u} = \int \left(\frac{2}{x} - 2\right)dx \]

\[ \ln|u| = 2\ln|x| - 2x + C \]

we can ignore +C and the absolute value signs, since we just want one solution

\[ y = ve^{2x} = c_1 e^{2x} + c_2 \left(\frac{1}{2} x^2 - \frac{1}{2} x - \frac{1}{4}\right) \]
4. Consider the following differential equation:

\[ y'' + 4y' + 13y = 10 \sin(3t) \]

(a) [8 pts.] Find all synchronous solutions of the form \( y = A \cos(3t) + B \sin(3t) \).

\[
\begin{align*}
y &= A \cos(3t) + B \sin(3t) \\
y' &= -3A \sin(3t) + 3B \cos(3t) \\
y'' &= -9A \cos(3t) - 9B \sin(3t)
\end{align*}
\]

\[
(-9A \cos(3t) - 9B \sin(3t)) + (-12A \sin(3t) + 12B \cos(3t)) \\
+ 13A \cos(3t) + 13B \sin(3t) = 10 \sin(3t)
\]

\[
\begin{align*}
-9A + 12B + 13A &= 0 \\
-9B - 12A + 13B &= 10
\end{align*}
\]

\[
\begin{align*}
-9A + 12B + 13A &= 0 \\
-9B - 12A + 13B &= 10
\end{align*}
\]

\[ \Rightarrow \quad A = -\frac{3}{4}, \quad B = \frac{1}{4} \]

\[ y = -\frac{3}{4} \cos(3t) + \frac{1}{4} \sin(3t) \]

(b) [12 pts.] Find all solutions to the differential equation.

**Homogeneous equation:** \( y'' + 4y' + 13y = 0 \)

\[ r^2 + 4r + 13 = 0 \]

**Matlab:** \( r = -2 \pm 3i \)

**Homogeneous solution:**

\[ y = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) \]

\[ y = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) - \frac{3}{4} \cos(3t) + \frac{1}{4} \sin(3t) \]
For the remaining two problems, you may use any MATLAB commands.

5. [12 pts.] For each of the following pairs of functions, determine whether the functions \( y_1 \) and \( y_2 \) are linearly independent for \( x > 0 \).

(a) \( y_1 = \ln(x^2) \)
    \( y_2 = \ln(x^3) \)

   \[ \ln(x^2) = 2 \ln(x) \]
   \[ \ln(x^3) = 3 \ln(x) \]

   \( y_1 = \frac{2}{3} y_2 \)

   **No, these are lin. dep.**

(b) \( y_1 = \ln(x) \)
    \( y_2 = \ln(3x) \)

   \( y_1(e) = \ln(e) = 1 \)
   \( y_2(e) = \ln(3e) = \ln(3) + 1 \)
   \( y_2(e^2) = \ln(e^2) = 2 \)
   \( y_2(e^2) = \ln(3e^2) = \ln(3) + 2 \)

   These are not multiples of each other.

   **Yes, these are lin. ind.**

(c) \( y_1 = \ln(3x) \)
    \( y_2 = 0 \)

   **No, these are lin. dep.**

   \[ 0(y_1) = y_2 \]
For the remaining problem you may use any MATLAB commands.

6. Consider the following mass-spring system, in which a block lying on a frictionless surface is attached to a wall by a spring:

Let \( x(t) \) be the distance of the block from its equilibrium position (measured in centimeters) at time \( t \) (measured in seconds). An external force \( F = \sin(2t) \) is applied to the mass-spring system, so that \( x(t) \) satisfies the following differential equation and initial conditions:

\[
x'' + 4x = \sin(2t), \quad x(0) = 1, \quad x'(0) = 0
\]

(a) [8 pts.] Use MATLAB to solve the differential equation with the given initial conditions.

```matlab
dsolve('D2*x + 4*x = sin(2*t)', 'x(0)=1', 'Dx(0)=0')
```

\[
x = \frac{1}{8} \sin(2t) + \cos(2t) - \frac{1}{4} \cos(2t)
\]

(b) [8 pts.] Describe the long-term behavior of the block. (In particular, how does the amplitude of the oscillations change over time?)

In the long-term, the block oscillates with increasing amplitude. The amplitude of the oscillation tends to \( \infty \) as \( t \to \infty \).