Name: Solutions

Directions:

1. You may use MATLAB during the test. This is the only program you may have open on the computer. You may not use any other computer programs, and you may not use a calculator.

2. For the first three problems, do not use MATLAB to solve the differential equations; in particular, you may not use the dsolve command (except to check your work). You may use MATLAB to compute integrals, perform arithmetic, and check your answers. You must show your work to receive any credit.

3. For the last three problems, you may use any MATLAB commands.

4. For all of the problems, show your work, and clearly indicate your final answer.

For instructor use only:

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<th>Points</th>
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<td>6</td>
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For the first three problems, do not use MATLAB to solve the differential equations; in particular, you may not use the `dsolve` command (except to check your work). You may use MATLAB to compute integrals, perform arithmetic, and check your answers. You must show your work to receive any credit.

1. [18 pts.] Solve the following initial value problem:

\[
\frac{dy}{dx} = e^{x+y}, \quad y(0) = \ln(2)
\]

This diff eq. is separable.
Since \( e^y > 0 \) for all \( y \), no solutions are lost during the separation.

\[
\int e^{-y} \, dy = \int e^x \, dx
\]

\[-e^{-y} = e^x + C\]

Initial condition \( x = 0, \) \( y = \ln(2) \):

\[-e^{-\ln(2)} = e^0 + C\]

\[-\frac{1}{2} = 1 + C \Rightarrow C = -\frac{3}{2}\]

\[-e^{-y} = e^x - \frac{3}{2}\]

\[e^{-y} = -e^x + \frac{3}{2}\]

\[-y = \ln|-e^x + \frac{3}{2}|\]

\[y = -\ln|-e^x + \frac{3}{2}|\]
2. [16 pts.] Solve the following differential equation:

\[
\frac{dr}{d\theta} + r \tan \theta = \sec \theta, \quad 0 < \theta < \pi/2
\]

This diff. eq. is linear.

\[
p(\theta) = \tan \theta \quad q(\theta) = \sec \theta
\]

\[
\mu(\theta) = e^{\int p d\theta} = e^{\int \tan \theta d\theta} = e^{-\ln|\cos \theta|}
\]

\[
= e^{\ln|\sec \theta|} = |\sec \theta| = \sec \theta
\]

Thus, \( \mu(\theta) = \sec \theta \)

\[
\sec \theta \frac{dr}{d\theta} + \sec \theta \tan \theta r = \sec^2 \theta
\]

\[
\frac{d}{d\theta} (r \sec \theta) = \sec^2 \theta
\]

\[
r \sec \theta = \int \sec^2 \theta d\theta
\]

\[
r \sec \theta = \tan \theta + C
\]

\[
r = \frac{\tan \theta + C}{\sec \theta}
\]

\[
r = \sin \theta + C \cos \theta
\]
3. [16 pts.] Solve the following differential equation:

\[
(2t \sin y + 2t) \, dt + (t^2 \cos y + 3e^y) \, dy = 0
\]

\[\text{M}(t,y) \quad \text{N}(t,y)\]

These are equal.

\[\frac{dM}{dy} = 2t \cos y\]

This diff. eq. is exact.

\[\frac{dN}{dt} = 2t \cos y\]

\[
\int M(t,y) \, dt = \int (2t \sin y + 2t) \, dt
\]

\[= t^2 \sin y + t^2 + g(y)\]

\[f(t,y)\]

\[
\frac{df}{dy} = (t^2 \cos y + g'(y))
\]

Compare

\[g'(y) = 3e^y\]

\[g(y) = \int 3e^y \, dy \]

\[= 3e^y + C\]

\[f(x,y) = t^2 \sin y + t^2 + 3e^y + C\]

\[\text{Solutions:} \quad t^2 \sin y + t^2 + 3e^y + C = 0\]
For the remaining problems, you may use any MATLAB commands.

4. [16 pts.] Consider the initial value problem

\[
\frac{dy}{dx} = f(y), \quad y(1) = 2
\]

where \( f \) is a function that takes the following values:

<table>
<thead>
<tr>
<th>( y )</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(y) )</td>
<td>-1.0</td>
<td>0.5</td>
<td>-1.8</td>
<td>-3.2</td>
<td>-3.7</td>
<td>-4.0</td>
<td>-4.2</td>
<td>-3.4</td>
<td>-2.1</td>
<td>-0.5</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Use Euler’s method with step size \( h = 0.1 \) to find an approximation for \( y(1.2) \).

\[
\begin{array}{c|c|c|c}
\hline
n & x_n & y_n & y_{n+1} = y_n + h f(x_n, y_n) \\
\hline
0 & 1 & 2 & \text{---} \\
1 & 1.1 & 1.6 & y_1 = y_1 + (0.1) f(2) \\
& & & = 2 + (0.1)(-4.0) \\
& & & = 2 - 0.4 = 1.6 \\
2 & 1.2 & 1.28 & y_2 = y_2 + (0.1) f(1.6) \\
& & & = 1.6 + (0.1)(-3.2) \\
& & & = 1.6 - 0.32 = 1.28 \\
\hline
\end{array}
\]

\[ y(1.2) \approx 1.28 \]
For the following problem, you may use any MATLAB commands.

5. [8 pts.] Consider the following initial value problem:

\[
\frac{dy}{dx} = x^2 + \sin x \cos y, \quad y(1) = 4
\]

Use `eul.m` with step size \( h = 0.001 \) to approximate \( y(2) \). Give your answer to six decimal places. (Note: For partial credit purposes, you may want to write the Matlab commands that you used.)

\[
\gg \text{format long}
\]
\[
\gg f = \text{inline}\left(\text{'x}^2 + \sin(x) \cdot \cos(y)', \text{'x'}, \text{'y'}}\right)
\]
\[
\gg [x, y] = \text{eul}\left(f, [1, 2], 4, 0.001\right);
\]
\[
\gg y(\text{size}(y,1))
\]

\[
y(2) \approx 6.357474
\]
For the following problem, you may use any MATLAB commands.

6. Consider the following differential equation:

\[
\frac{dy}{dx} = y^2 + c
\]

where \( c \) is a constant.

(a) [6 pts.] What are the constant solutions when \( c < 0 \)? What are the constant solutions when \( c > 0 \)?

For \( c < 0 \):

\[
\begin{align*}
[c < 0] \\
y^2 + c &= 0 \\
y^2 &= -c \\
y &= \pm \sqrt{-c}
\end{align*}
\]

For \( c > 0 \):

\[
\begin{align*}
[c > 0] \\
y^2 + c &= 0 \\
in no constant solutions
\end{align*}
\]

(b) [8 pts.] Suppose \( c < 0 \). Draw the phase line for this differential equation. Clearly mark the equilibrium points. For each equilibrium indicate whether it is a source, a sink, or neither.

- **Source** \( y = \sqrt{-c} \)
- **Sink** \( y = -\sqrt{-c} \)
\[ \frac{dy}{dx} = y^2 + c \]

(c) [6 pts.] Suppose \( c > 0 \). Draw the phase line for this differential equation. Clearly mark the equilibrium points. For each equilibrium indicate whether it is a source, a sink, or neither.

\[ \lim_{{x \to \infty}} y(x) ? \]

(d) [3 pts.] If \( c = -2 \) and \( y(0) = 1 \), what is \( \lim_{{x \to \infty}} y(x) \)?

\[ \lim_{{x \to \infty}} y(x) = -\sqrt{2} \]

(e) [3 pts.] If \( c = -2 \) and \( y(0) = -3 \), what is \( \lim_{{x \to \infty}} y(x) \)?

\[ \lim_{{x \to \infty}} y(x) = -\sqrt{2} \]