Math 308 Practice Test 2A

For the first two problems, do not use MATLAB to solve the differential equations; in particular, you may not use the `dsolve` command (except to check your work). You may use MATLAB to compute integrals, perform arithmetic, and check your answers. You must show your work to receive any credit.

1. Solve the following differential equation:

\[(x - y + 1)\,dx + (2x - 2y + 4)\,dy = 0\]

You may leave your answer in implicit form.

Since \(x - y\) and \(2x - 2y\) are multiples of each other, we cannot use the method of linear coefficients. Instead, we make the substitution: \(z = x - y\)

\[
\frac{dz}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}
\]

The diff. eq. becomes:

\[(z + 1) + (2z + 4)\left(1 - \frac{dz}{dx}\right) = 0\]

\[-\frac{dz}{dx} = \frac{-z + 1}{2z + 4}\]

\[
\frac{dz}{dx} = 1 + \frac{z + 1}{2z + 4}
\]

\[
\frac{dz}{dx} = \frac{2z + 4 + z + 1}{2z + 4}
\]

\[
\frac{dz}{dx} = \frac{3z + 5}{2z + 4}
\]

\[
\int \left(\frac{2z + 4}{3z + 5}\right)\,dz = \int dx
\]

\[
\frac{2}{3}z + \frac{2}{9} \ln|3z + 5| = x + C
\]

(found using MATLAB)

\[
\frac{2}{3}(x - y) + \frac{2}{9} \ln|3x - 3y + 5| = x + C
\]

\[
-\frac{1}{3}x - \frac{2}{3}y + \frac{2}{9} \ln|3x - 3y + 5| = C
\]
2. Solve the following differential equation:

\[ \frac{dr}{d\theta} - r \tan \theta = r^2 \sec \theta \]

This diff. eq. is Bernoulli.

Divide by \( r^2 \):

\[ \frac{1}{r^2} \frac{dr}{d\theta} - \frac{\tan \theta}{r} = \sec \theta \]

Substitution:

\[
\begin{aligned}
  v &= \frac{1}{r} = r^{-1} \\
  \frac{dv}{d\theta} &= -r^{-2} \frac{dr}{d\theta} \\
  &= \frac{1}{r^2} \frac{dr}{d\theta}
\end{aligned}
\]

This diff. eq. is linear.

\[ \mu(\theta) = e^{\int \tan \theta \, d\theta} = e^{\ln|\cos \theta|} = |\sec \theta| = \sec \theta \]

\[ \sec \theta \frac{dv}{d\theta} + \sec \theta v \tan \theta = -\sec^2 \theta \]

\[ \frac{d}{d\theta} (v \sec \theta) = -\sec^2 \theta \]

\[ v \sec \theta = \int -\sec^2 \theta \, d\theta \]

\[ v \sec \theta = -\tan \theta + C \]

\[ v = -\sin \theta + C \cos \theta \]

\[ \frac{1}{r} = -\sin \theta + C \cos \theta \]

Add to problem:

\[ 0 \leq \theta \leq \frac{\pi}{2} \]
For the remaining three problems, you may use any MATLAB commands. All answers should either be exact or correct to 4 decimal places.

3. A petri dish initially contains 100 bacteria. After 1 hour, the petri dish contains 200 bacteria. The bacteria population grows at a rate proportional to the current population (that is, the bacteria population satisfies the Malthusian population model).

(a) How many bacteria are in the petri dish after $t$ hours?

$$\frac{dP}{dt} = kP, \quad P(0) = 100$$

Solution (MATLAB):

$$P = 100e^{kt}$$

Since $P(1) = 200$, $200 = 100e^k$

$$\Rightarrow \quad k = \ln(2)$$

$$P(t) = 100e^{\ln(2)t}$$

$$P(t) = 100(2^t)$$

(b) How many bacteria are in the petri dish after 1 day?

$$P(24) = \left[100 \left(2^{24}\right)\right]_{\text{bacteria}}$$

$$\approx 1.6777 \times 10^9$$

either answer would be accepted
4. A cannonball with mass 8 kg is fired directly upwards with initial velocity 20 m/s, and then allowed to fall under the influence of gravity. Assume that the force in Newtons due to air resistance is $-16v$, where $v$ is the velocity of the cannonball in m/s.

(a) Determine the equation of motion of the cannonball.

Solution (Matlab):

\[
v = -4.9 + 24.9e^{-2t}
\]

\[
x(t) = \int v(t) \, dt
\]

\[= \int (-4.9 + 24.9e^{-2t}) \, dt
\]

\[= -4.9t - 12.45e^{-2t} + C
\]

Since $x(0) = 0$,

\[C = 12.45
\]

The position of the ball is

\[x(t) = -4.9t - 12.45e^{-2t} + 12.45
\]

meters above the ground

Note: These decimals are exact.

For parts (b), (c), and (d), it will be helpful to have $v(t)$ and $x(t)$ stored in Matlab:

\[
\gg \text{syms t}
\]

\[
\gg x = -4.9 * t - 12.45 * \exp(-2*t) + 12.45
\]

\[
\gg v = -4.9 + 24.9 * \exp(-2 * t)
\]
(b) When will the cannonball reach its maximum height?

The cannonball will reach its max height when \( v = 0 \).

\[
\Rightarrow \text{solve } (v, 't') \\
\Rightarrow \text{eval } (\text{ans})
\]

\( t = 0.8128 \text{ seconds} \)

(c) What is the maximum height reached?

\[
\Rightarrow \text{subs } (x, 't', \text{ans})
\]

\( 6.0172 \text{ meters} \)

(d) When will the cannonball strike the ground (assuming that it was initially fired from the ground)?

\[
\Rightarrow \text{solve } (x, 't') \\
\Rightarrow \text{eval } (\text{ans})
\]

\( 2.5245 \text{ seconds} \)
5. A tank contains 2000 liters of salt water with an initial concentration of 0.05 kg/L. Salt water enters the tank at a rate of 100 L/day. The concentration $c$ of salt in the entering salt water is determined by the following equation:

$$c = 0.04 + 0.02 \cos(2\pi t/7)$$

with $c$ measured in kg/L and $t$ measured in days, with $t = 0$ occurring at noon on a Monday. The solution in the tank is kept thoroughly mixed and salt water drains from the tank at the same rate that it enters (100 L/day).

(a) Set up a differential equation that describes the amount of salt in the tank at time $t$.

$$y(t) = \text{amount of salt in tank after } t \text{ days}$$

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\frac{dy}{dt} = 100(0.04 + 0.02 \cos(2\pi t/7)) - \frac{y}{2000}(100)$$

$$\frac{dy}{dt} = 4 + 2 \cos(2\pi t/7) - \frac{y}{20}$$

(b) When during the week is the concentration of salt entering the tank the greatest?

$c$ is maximized when $t = 0, 7, 14, 21, \text{ etc.}$

Thus the concentration is maximized at noon every Monday.

(c) During the second week, on what day of the week is the concentration of salt in the tank maximized?

Tuesday

To see this either

- use dsolve
- use fieldplot