Math 308 Practice Test 1B

For the first four problems, do not use MATLAB to solve the differential equations; in particular, you may not use the `dsolve` command (except to check your work). You may use MATLAB to compute integrals, perform arithmetic, and check your answers. You must show your work to receive any credit.

1. Determine whether \( x^3 + \sin(x^2y) = 3 \) is an implicit solution to the following differential equation

\[
\frac{y'}{x} = 3x \sec(xy) + \frac{y}{x}
\]

Implicit diff:

Method 1:

\[
3x^2 + \cos(x^2y)(2xy + x^2 \frac{dy}{dx}) = 0
\]

\[
x \cos(x^2y) \frac{dy}{dx} = -3x^2 - 2xy \cos(x^2y)
\]

\[
\frac{dy}{dx} = \frac{-3x^2 - 2xy \cos(x^2y)}{x^2 \cos(x^2y)} = -3 \sec(x^2y) - \frac{2y}{x}
\]

These are not equal. Note for example that they are not equal when \( x = 1 \) and \( y = 0 \).

No, it is not a solution to the diff. eq.
2. Solve the following initial value problem.

\[ y \cos(xy) \, dx + (x \cos(xy) + 2) \, dy = 0, \quad y(0) = 1 \]

This diff. eq. is exact:

\[ \frac{\partial M}{\partial y} = \cos(xy) - xy \sin(xy) \]
\[ \frac{\partial N}{\partial x} = \cos(xy) - xy \sin(xy) \]

These are equal.

\[ \int M \, dx = \int y \cos(xy) \, dx = \frac{\sin(xy)}{f(x,y)} + g(y) \]

Comparing with \( N \), we see that

\[ g'(y) = 2 \]

\[ g(y) = \int 2 \, dy = 2y + C \]

\[ f(x,y) = \sin(xy) + 2y + C \]

Solution: \( \sin(xy) + 2y + C = 0 \)

initial condition \( y(0) = 1 \) \( (x=0, y=1) \)

\( \sin(0 \cdot 1) + 2 \cdot 1 + C = 0 \Rightarrow C = -2 \)

The solution is \( \boxed{\sin(xy) + 2y - 2 = 0} \)
3. Solve the following differential equation:

\[ \frac{dr}{d\theta} = r \sec \theta \]

This diff eq. is separable:

\[ \int \frac{dr}{r} = \int \sec \theta \, d\theta \]

\[ \ln |r| = \ln |\sec \theta + \tan \theta| + C \]

\[ |r| = e^C \sec \theta + \tan \theta \]

\[ r = \pm e^C |\sec \theta + \tan \theta| \]

OK, \( r = 0 \) is included in this general solution.

\[ r = A(\sec \theta + \tan \theta) \]
4. Solve the following differential equation:

\[ xy' + 2y = \sin x \]

This diff. eq. is first order linear.

\[ y' + \left( \frac{2}{x} \right) y = \frac{\sin x}{x} \]

\[ \uparrow \quad \uparrow \]

\[ p(x) \quad q(x) \]

integrating factor:

\[ \frac{2}{x} \int dx = 2 \ln x = 2 \ln(x^2) = x^2 \]

\[ \mu(x) = e^{\int \frac{2}{x} \, dx} = e^{2 \ln x} = e^{\ln(x^2)} = x^2 \]

multiply diff eq. by \[ \mu(x) = x^2 \]

\[ x^2 y' + 2xy = x \sin x \]

\[ \frac{d}{dx} (x^2 y) = x \sin x \]

\[ x^2 y = \int x \sin x \, dx \]

\[ x^2 y = -x \cos x + \sin x + C \]

\[ y = \frac{-x \cos x + \sin x + C}{x^2} \]
For the remaining 3 problems, you may use any MATLAB commands.

5. Consider the initial value problem

\[ y' = \sin(x) \sin(y), \quad y(0) = 1 \]

Use `eul.m` with step size \( h = .001 \) to find an approximation for \( y(1) \).

\[
y(1) \approx 1.42595670085213
\]

6. Consider the initial value problem

\[ y' = 2\sqrt{ax + by}, \quad y(0) = 1 \]

where \( a \) and \( b \) are constants. Use Euler’s method with stepsize \( 1/2 \) to find an approximation for \( y(1) \) (your answer should be in terms of \( a \) and \( b \)).

\[
\begin{array}{|c|c|c|}
\hline
n & x_n & y_n \\
\hline
0 & 0 & 1 \\
1 & \frac{1}{2} & 1 + \sqrt{b} \\
2 & 1 & 1 + \sqrt{b} + \sqrt{\frac{a}{2} + b + b \sqrt{b}} \\
\hline
\end{array}
\]

\[
y_{n+1} = y_n + \left( \frac{1}{2} \right) 2\sqrt{ax_n + by_n}
\]

\[
y_1 = 1 + \left( \frac{1}{2} \right) \left( 2 \right) \sqrt{a \cdot 0 + b \cdot 1}
    = 1 + \sqrt{b}
\]

\[
y_2 = 1 + \sqrt{b} + \sqrt{a \cdot \frac{1}{2} + b \cdot (1 + \sqrt{b})}
    = 1 + \sqrt{b} + \sqrt{\frac{a}{2} + b + b \sqrt{b}}
\]

\[
y(1) \approx 1 + \sqrt{b} + \sqrt{\frac{a}{2} + b + b \sqrt{b}}
\]
7. Consider the following differential equation:

\[ \frac{dy}{dt} = (1/5)(y + 1)(y - 2)^2(y - 4) \]

(a) Draw the phase line for this differential equation. Clearly indicate all of the equilibria. For each equilibrium, note whether it is a sink, a source, or neither.

![Phase line diagram]

(b) If \( y(t) \) is a solution with initial condition \( y(0) = 1 \), what is \( \lim_{t \to \infty} y(t) \)?

\[ \lim_{t \to \infty} y(t) = -1 \]

(c) If \( y(t) \) is a solution with initial condition \( y(0) = 5 \), what is \( \lim_{t \to \infty} y(t) \)?

\[ y(t) \to \infty \]

(This is actually a bad question—Sorry.)

The limit is undefined because \( y(t) \) has vertical asymptotes, but you can’t tell that from the phase line.)