Math 308 Practice Final A

For the first seven problems, do not use MATLAB to solve the differential equations, compute inverse Laplace transforms, or to compute eigenvalues and eigenvectors; in particular, you may not use the commands dsolve, ilaplace, or eig (except to check your work). You may use MATLAB to compute integrals, solve equations, compute Laplace transforms, perform row reduction, take determinants, perform arithmetic, and check your answers. You must show your work to receive any credit.

1. Solve the following differential equation:

\[ \frac{dx}{dt} = \cosh(t)\sqrt{x+1} \]

This diff. eq. is separable.

\[ \int \frac{dx}{\sqrt{x+1}} = \int \cosh(t) \, dt \]

\[ 2\sqrt{x+1} = \sinh(t) + C \]

\[ \sqrt{x+1} = \frac{1}{2} \sinh(t) + C \]

\[ x+1 = \left( \frac{1}{2} \sinh(t) + C \right)^2 \]

\[ x = \left( \frac{1}{2} \sinh(t) + C \right)^2 - 1 \]

or \[ x = \left( \frac{1}{4} e^t - \frac{1}{4} e^{-t} + C \right)^2 - 1 \]
2. For the following differential equations, determine whether the method of undetermined coefficients can be used to find a particular solution. If it can, determine the form of a particular solution. You do not need to solve the differential equation.

(a) \( y'' + 9y = t^2 \sin(3t) + e^{2t} \)

\[ r^2 + 9 = 0 \]
\[ r = \pm 3i \]
\[ y_h = c_1 \cos(3t) + c_2 \sin(3t) \]

Yes.

\[ y_p = (At^3 + Bt^2 + Ct) \sin(3t) + (Dt^3 + Et^2 + Ft) \cos(3t) + G e^{2t} \]

(b) \( y'' + 4y = e^{-t} \cos(3t) \)

No.
(c) \( y'' - 4y' + 4y = \sinh^2 t \Rightarrow \left( \frac{1}{2} e^t - \frac{1}{2} e^{-t} \right)^2 \)
\[ r^2 - 4r + 4 = 0 \]
\[ (r - 2)^2 = 0 \]
\[ y_h = c_1 e^{2t} + c_2 t e^{2t} \]

\[ \boxed{y_p = A + Bt e^{2t} + C e^{-2t}} \]

Yes

(d) \( y'' + 5y' + 6y = \ln t \)

\[ \boxed{\text{No}} \]

(e) \( y'' + 3y' + 2y = t \sin(2t) \cos(2t) \)
\[ r^2 + 3r + 2 = 0 \]
\[ r = -2, -1 \]
\[ y_h = c_1 e^{-2t} + c_2 e^{-t} \]
\[ y_p = (At + B) \sin(4t) + (Ct + D) \cos(4t) \]
3. Consider the following differential equation:

\[ 2t^2y'' + 3ty' - y = 0 \]

One solution to this differential equation is \( f(t) = \frac{1}{t} \). Find all solutions to the differential equation.

**Reduction of Order**

\[
\begin{align*}
y &= v(t) \cdot \frac{1}{t} = \frac{v}{t} = vt^{-1} \\
y' &= v' t^{-1} - v t^{-2} \\
y'' &= v'' t^{-1} - v' t^{-2} - v t^{-3} + 2v t^{-3} \\
    &= v'' t^{-1} - 2v' t^{-2} + 2t^{-3}
\end{align*}
\]

**Plug in to diff. eq.:**

\[
2t^2 \left( v'' t^{-1} - 2v' t^{-2} + 2v t^{-3} \right) + 3t \left( v' t^{-1} - v t^{-2} \right) - v t^{-1} = 0
\]

\[
v''(2t) + v'(-4 + 3) + v(4t^{-1} - 3t^{-1} - t^{-1}) = 0
\]

\[
2tv'' - v' = 0 \\
2tu' - u = 0 \Rightarrow \int \frac{2du}{u} = \int \frac{dt}{t} \Rightarrow 2\ln|u| = \ln|t| + C
\]

\[
\Rightarrow u = \sqrt{t}
\]

\[
v = \int \sqrt{t} \, dt = \frac{2}{3} t^{3/2} + C
\]

**Second lin. ind. solution:**

\[
y = v t^{-1} = \frac{2}{3} \sqrt{t}
\]

Since we are multiplying by \( c_2 \), the \( \frac{2}{3} \) doesn't matter.

**General solution**

\[
y = \frac{c_1}{t} + c_2 \sqrt{t}
\]
4. Solve the following initial value problem (don’t use the ilaplace command):

\[ y'' + 4y' + 5y = 2\delta(t - 3), \quad y(0) = 0, y'(0) = 1 \]

\[ L \{ y'' + 4y' + 5y \} = L \{ 2\delta(t - 3) \} \]

\[ s^2 L \{ y \} - sy(0) - y'(0) + 4s L \{ y \} - y(0) = 2e^{-3s} \]

\[ (s^2 + 4s + 5)L \{ y \} - 1 = 2e^{-3s} \]

\[ L \{ y \} = \frac{2e^{-3s}}{s^2 + 4s + 5} + \frac{1}{s^2 + 4s + 5} \]

\[ L^{-1} \left\{ \frac{1}{s^2 + 4s + 5} \right\} = L^{-1} \left\{ \frac{1}{(s+2)^2 + 1} \right\} = e^{-2t} \sin t \]

\[ = \frac{(s^2 + 4s + 4) - 4s}{(s^2 + 4s + 4) - 4s + 5} \]

\[ L^{-1} \left\{ \frac{2e^{-3s}}{s^2 + 4s + 5} \right\} = 2e^{-2(t-3)} \sin(t-3) \cdot u(t-3) \]

\[ y = 2e^{-2t} + 6 \sin(t-3) \cdot u(t-3) + e^{-2t} \sin t \]
5. Find a power-series expansion about $x = 0$ for a general solution to the following differential equation. Write your answer using summation notation.

$$2y'' + xy' + y = 0$$

$$y = q_0 + q_1 x + q_4 x^2 + q_3 x^2 + q_4 x^4 + \ldots + \sum_{n=1}^{\infty} q_n x^n$$

$$y' = q_1 + 2q_2 x + 3q_3 x^2 + 4q_4 x^3 + \ldots + \sum_{n=1}^{\infty} q_n x^n$$

$$y'' = 2q_2 + 3q_3 x + 4q_4 x^2 + \ldots + \sum_{n=1}^{\infty} (n-1) q_n x^{n-1}$$

$$xy' = q_1 x + 2q_2 x^2 + 3q_3 x^3 + 4q_4 x^4 + \ldots + \sum_{n=1}^{\infty} n q_n x^n$$

$$2y'' + xy' + y = 0$$

$$(2.2q_2 + q_0) + (2.2.3q_3 + q_1 + q_0) x + (2.2.3.4q_4 + 2q_2 + q_2) x^2 + \ldots + (2(n+1)(n+2)q_{n+2} + nq_{n+1} + q_0) x^n = 0$$

$q_0$ and $q_1$ are constants

$$2.2q_2 + q_0 = 0 \Rightarrow q_2 = -\frac{q_0}{2.2}$$

$$2.2.3q_3 + 2q_1 = 0 \Rightarrow q_3 = -\frac{q_1}{2.3}$$

$$2.2.3.4q_4 + 2q_2 + q_2 = 0 \Rightarrow q_4 = -\frac{3q_2}{2.3.4} = \frac{q_0}{2.2.2.4.6} = \frac{q_0}{2.2.2.4.6}$$

$$a_5 = \frac{q_1}{2.2.3.5} = \frac{q_1}{2.2.3.5}$$

$$a_6 = \frac{a_5}{2.2.3.4.6} = \frac{-q_1}{2.2.3.5.7}$$

$$a_7 = \frac{-q_1}{2.2.3.5.7}$$

$$y = q_0 + q_1 x - \frac{q_0}{2.2} x^2 - \frac{q_1}{2.3} x^3 + \frac{q_0}{2.2.4} x^4 + \frac{q_1}{2.2.3.5} x^5 - \frac{q_0}{2.2.3.5.7} x^6 - \frac{q_1}{2.2.3.5.7} x^7 + \ldots$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n q_0 x^{2n}}{2^n (2.4.6\ldots2n)} + \sum_{n=0}^{\infty} \frac{(-1)^n q_1 x^{2n+1}}{2^n (3.5.7\ldots(2n+1))}$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n q_0 x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n q_1 x^{2n+1}}{2^n (2n+1)!}$$
6. Solve the following system of differential equations. Don’t use the Matlab commands eig or dsolve, but you may use Matlab to perform row reduction or take determinants.

\[
x' = 5x - 5y - 5z \\
y' = -x + 4y + 2z \\
z' = 3x - 5y - 3z
\]

\[
\det \begin{pmatrix} 5-\lambda & -5 & -5 \\ -1 & 4-\lambda & 2 \\ 3 & -5 & -3-\lambda \end{pmatrix} = 10 - 13\lambda + 6\lambda^2 - \lambda^3 = 0
\]

Solve: \( \lambda = 2, 2 \pm i \)

\[
\lambda = 2
\]

\[
\begin{pmatrix} 3 & -5 & -5 & 0 \\ -1 & 2 & 2 & 0 \\ 3 & -5 & -5 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y+2z \\ z-t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\lambda = 2 + i
\]

\[
\begin{pmatrix} 3-i & -5 & -5 & 0 \\ -1 & 2-i & 2 & 0 \\ 3 & -5 & -5-i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x-2z \\ y+(4+2i)z \\ x+i \\ y+2i \end{pmatrix} = 0
\]

Multiply by \( S \):

\[
\begin{pmatrix} x \\ y \\ z-t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
\]

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{-it} \begin{pmatrix} \cos t \cdot \frac{5}{2} \\ -\sin t \cdot \frac{5}{2} \end{pmatrix} + e^{2t} \begin{pmatrix} \sin t \cdot \frac{5}{2} \\ \cos t \cdot \frac{5}{2} \end{pmatrix}
\]
7. Consider the initial value problem

\[ \frac{dy}{dx} = f(y), \quad y(0) = 1 \]

where \( f \) is a function that takes the following values:

<table>
<thead>
<tr>
<th>( y )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(y) )</td>
<td>-2.0</td>
<td>-1.3</td>
<td>-2.4</td>
<td>-1.8</td>
<td>-1.6</td>
<td>-2.0</td>
<td>-1.3</td>
<td>-0.8</td>
<td>-0.5</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Use Euler's method with step size \( h = 0.1 \) to find an approximation for \( y(0.2) \).

\[
\begin{array}{c|c|c|c|}
 n & x_n & y_n & y_{n+1} \\
\hline
 0 & 0 & 1 & y_1 = 1 + (0.1)(-2) = \frac{8}{10} \\
 1 & 0.1 & 0.8 & y_2 = 0.8 + (0.1)(-1.6) = \frac{64}{100} \\
 2 & 0.2 & 0.64 & \\
\end{array}
\]

\[ y(0.2) \approx 0.64 \]
For the remaining four problems, you may use any MATLAB commands.

8. Consider the recursively defined sequence with \( a_0 = 0.5 \), \( a_1 = 0.2 \), and \( a_{n+1} = a_n^2 - a_{n-1} \).
   Find \( a_{100} \) to 6 decimal places.

```matlab
format long
a = 0.5;
b = 0.2;
for i = 1:99
    c = b^2 - a;
a = b;
b = c;
end;
c
```

\[ a_{100} = -0.062598 \]
9. A one loop RLC-circuit consists of a 2-Ω resistor, an inductor with \( L = 0.25 \) Henrys, a capacitor with \( C = 0.1 \) Farads, and an AC generator that produces a voltage of \( E(t) = 40 \cos 2t \) volts. Set up a differential equation for the current \( I \) in the circuit. (You do not need to solve the differential equation.)

\[
\begin{align*}
E_R &= R I \\
E_L &= L \frac{dI}{dt} \\
E_C &= \frac{1}{C} q \\
(I &= \frac{dq}{dt})
\end{align*}
\]

\[
40 \cos (2t) = 2I + 0.25 \frac{dI}{dt} + \frac{q}{0.1}
\]

\[
40 \cos (2t) = 0.25 \frac{d^2 q}{dt^2} + 2 \frac{dq}{dt} + 10q
\]

**Differentiate:**

\[
0.25 \frac{d^3 q}{dt^2} + 2 \frac{d^2 I}{dt^2} + 10 \frac{dI}{dt} = -80 \sin (2t)
\]

\[
0.25 \frac{d^2 I}{dt^2} + 2 \frac{dI}{dt} + 10I = -80 \sin (2t)
\]
10. Consider the following differential equation:

\[
\frac{dy}{dx} = a - by
\]

where \( a \) and \( b \) are constants with \( a > 0 \) and \( b > 0 \).

(a) Determine the equilibrium solutions to the differential equation.

\[ q - by = 0 \Rightarrow y = \frac{a}{b} \]

(b) Sketch the phase line for the differential equation.

(c) What is \( \lim_{x \to \infty} y(x) \)?

\[ \lim_{x \to \infty} y(x) = \frac{a}{b} \quad (\text{from the phase line}) \]
11. A tank initially contains 100 L of pure water. Salt water with a concentration of 0.5 kg/L enters the tank at a rate of 2 L/min. The solution in the tank is kept thoroughly mixed and is drained from the tank at the same rate (2 L/min). After 10 minutes, the process is stopped, and fresh water is poured in at a rate of 2 L/min, with the mixture being drained at 2 L/min.

(a) Set up a differential equation describing the amount of salt in the tank after $t$ minutes (from when the salt water begins entering the tank). You will want to use a piecewise continuous function in the differential equation.

\[
\text{rate in} = \begin{cases} 
0.5 \times 2 & \text{if } t < 10 \\
0 & \text{if } t > 10 
\end{cases} = 1 + (0.1)(t-10)
\]

\[
\text{rate out} = \frac{dx}{100} = \frac{x}{50}
\]

\[
\frac{dx}{dt} = 1 - u(t-10) - \frac{x}{50}
\]

\[
x(0) = 0
\]

(b) How much salt is in the tank after 20 minutes?

\[
\text{dsolve} \left( 'Dx = 1 - \text{heaviside}(t-10) - \frac{x}{50} ' \right), \quad x(0) = 0, \quad t \left( 0^0, 20 \right)
\]

\[
\text{subs} \left( \text{ans}, t, 20 \right)
\]

\[
7.4205 \text{ kg}
\]

Exact answer: \(50e^{-1/5} - 50e^{-2/5} \text{ kg}\)