Research Statement

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1 Introduction

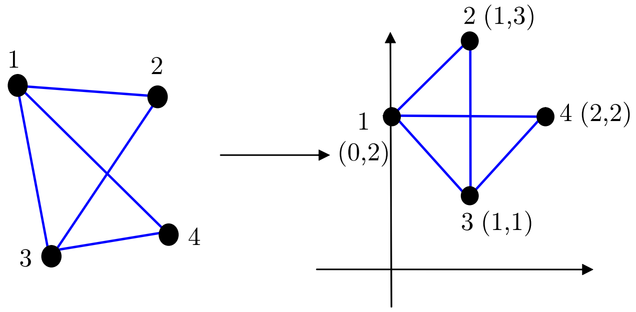
My research interests are discrete geometry, computational geometry, and graph theory. I particularly like problems that involve both graphs and geometry. In my research, I have explored two main problems: realizability of graphs and the Kneser-Poulsen conjecture. There remain many open questions related to both of these problems, and I intend to continue working on these questions, and also to explore other areas of discrete and computational geometry.

In addition, I would like to become involved in research with undergraduates. Many problems from discrete and computational geometry seem ideal for undergraduate research problems. The questions are accessible to undergraduates, as they generally require only a minimal amount of background to understand. Also, since they usually involve concrete objects (either graphs or objects in Euclidean space), it is possible to experiment with examples by hand or with a computer.

2 Realizability of Graphs

A *graph* G is a finite set of vertices $V(G) = \{1, \ldots, n\}$ and a finite set of edges $E(G)$, where each edge is a set containing exactly two vertices. The graphs we consider do not contain loops or multiple edges.

A *realization* of a graph G is a placement of the vertices into $\mathbb{R}^d$:



We think of the edges of a realization as straight line segments between the vertices. We allow multiple vertices in a realization to be placed to the same point, and edges may intersect or even overlap.

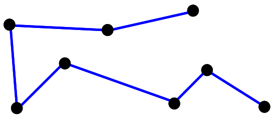
My advisor Robert Connelly introduced the following definition:

Definition 1. *A graph G is d-*realizable *if, given any realization of the graph in some finite dimensional Euclidean space, there exists a realization in $\mathbb{R}^d$ with the same edge lengths.*

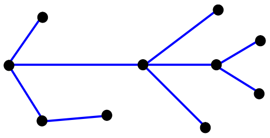
An alternative characterization: if we are given edge lengths for a d-realizable graph such that the graph can be realized in some high dimensional Euclidean space (that is, the edge lengths satisfy the triangle inequality, and higher dimensional generalizations of the triangle inequality), then the graph can be realized in $\mathbb{R}^d$. For $d = 1, 2$ and 3, this turns out to be equivalent to saying that a graph is d-realizable if *every* realization of the graph can be moved into $\mathbb{R}^d$ without changing the edge lengths.

Examples.

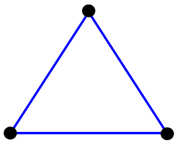
1. A path is 1-realizable, because we can rearrange the vertices in order on a line with the appropriate distance between the points.



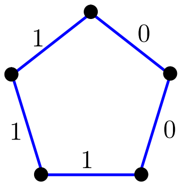
2. Similarly, a tree (a connected graph with no cycles) is also 1-realizable. (It is okay for edges to overlap.)



3. The triangle is 2-realizable, but not not 1-realizable, because the triangle with all edge lengths 1 can be realized in $\mathbb{R}^2$ but not in $\mathbb{R}^1$.



4. Similarly, the n-gon (for $n \geq 3$) is 2-realizable, but not 1-realizable, because the n-gon with three edges of length 1 and all remaining edges of length 0 can be realized in $\mathbb{R}^2$, but not in $\mathbb{R}^1$.



Based on the above examples, a connected graph is 1-realizable if and only if it is a tree. It turns out [BC07a] that a graph is 2-realizable if and only if it is series parallel (a term from graph theory).

A *minor* of a graph G is any graph obtained from G by a sequence of edge deletions and edge contractions.

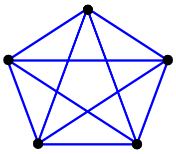
Theorem 1 (Connelly)**.** *If a graph G is d-realizable and H is a minor of G, then H is d-realizable.*

This means that d-realizability is a *minor monotone graph property*, so the Graph Minor Theorem of Robertson and Seymour [RS04] applies:

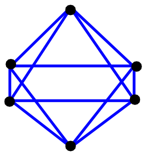
Theorem 2 (The Graph Minor Theorem)**.** *Every minor monotone graph property has a finite list of forbidden minors; i.e. there exits a finite list of graphs $G_1, \ldots, G_n$ such that a graph G satisfies the graph property if and only if G does not have any G_i as a minor.*

Connelly and I classified 3-realizable graphs in terms of forbidden minors.

Theorem 3 (Belk and Connelly)**.** *The forbidden minors for* 3*-realizable graphs are* K_5 *and the* 1*-skeleton of the octahedron.*

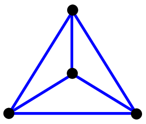


K_5

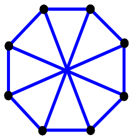


Octahedron

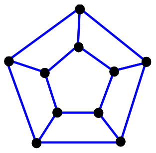
In addition, Connelly and I showed that every 3-realizable graph can be constructed from the graphs K_4, V_8, and $C_5 \times C_2$.



K_4



V_8



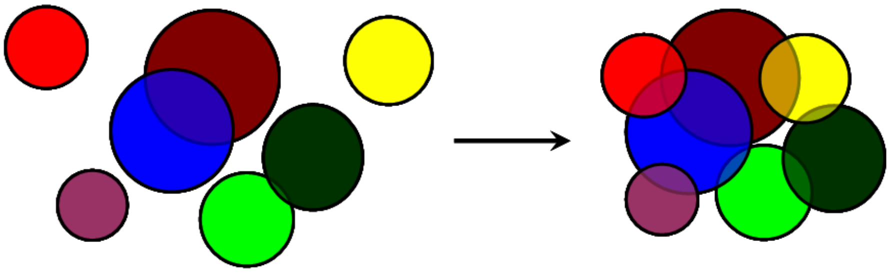
$C_5 \times C_2$

Future Plans: The obvious next question to ask is how to classify 4-realizable graphs. Unfortunately, this appears to be a relatively hard question to answer. There is an obvious class of graphs to look for forbidden minors to 4-realizability: the forbidden minors for graphs with treewidth at most 4 (forbidden minors for treewidth at most 3 were used in classifying 3-realizability). There are over 75 such forbidden minors, but knowing which of these are 4-realizable could provide some insight into the question. Additionally, it may be possible to find an nice algorithm to determine whether a graph is 4-realizable without finding the forbidden minors. This would be analogous with treewidth at most 4 — there is a linear time algorithm to determine whether a graph has treewidth at most 4 [Sa96], but the complete list of forbidden minors is unknown.

The motivation for d-realizability is the following problem: given a graph and specified edge lengths, find a realization in $\mathbb{R}^3$ (or $\mathbb{R}^d$) if one exists. Answers to this question have many possible applications — for example, if a chemist determines the distances between some of the atoms in a molecule, the chemist would like to create possible realizations of the molecule based on the data. Unfortunately, determining whether a graph with specified edge lengths has a realization in $\mathbb{R}^3$ is an NP-complete problem. The result of Theorem 3 is that for a very specific class of graphs, we can quickly determine whether the graph has a realization in $\mathbb{R}^3$. It would be interesting to find other such classes of graphs. For example, is there a general way to quickly determine whether 4-realizable graph with specified edge lengths can be realized in $\mathbb{R}^3$?

3 Kneser-Poulsen Conjecture

Consider a collection of possibly overlapping balls in d-dimensional Euclidean space. Suppose the balls are rearranged so that the distances between the centers of the balls all decrease or stay the same:



Kneser [Kn55] and Poulsen [Po54] independently conjectured in the 1950's that the volume of the union must either decrease or stay the same. Bezdek and Connelly [BC02] proved the conjecture for 2-dimensional Euclidean space, but it is still unknown for higher dimensions. The proof of Bezdek and Connelly uses the following well-known lemma (see [BC02], [CP91], and [Gr87]):

Lemma 1. *Suppose* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *are configurations of points in* n*-dimensional Euclidean space, and suppose that* $\mathbf{q}$ *is a contraction of* $\mathbf{p}$ *(meaning that* $d(q_i, q_j) \leq d(p_i, p_j)$ *for all* i *and* j*). Then there exists a continuous contraction from* $\mathbf{p}$ *to* $\mathbf{q}$ *in dimension dimension* $2n$ *(meaning there is a continuous motion from* $\mathbf{p}$ *to* $\mathbf{q}$ *in which the distances change monotonically).*

Bezdek and Connelly [BC02] use the above lemma to prove the Kneser-Poulsen conjecture in dimension 2 by proving the following theorem.

Theorem 4 (Bezdek and Connelly)**.** *Suppose that* $B(p_i)$ *and* $B(q_i)$ *are balls centered at* p_i *and* q_i*. If* $\mathbf{q} = (q_1, \ldots, q_n)$ *is a contraction of* $\mathbf{p} = (p_1, \ldots, p_n)$ *in* n*-dimensional Euclidean space and there exists a continuous contraction from* $\mathbf{p}$ *to* $\mathbf{q}$ *in* $n + 2$ *dimensions, then*

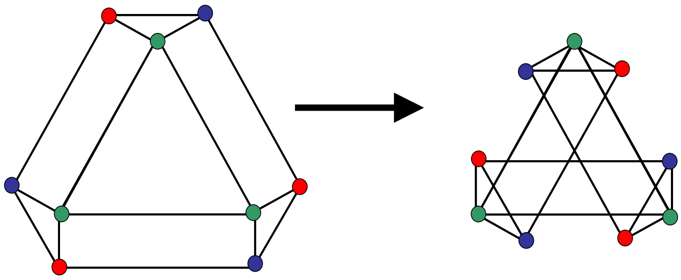
$$\mathrm{Vol}\left(\bigcup B(p_i)\right) \leq \mathrm{Vol}\left(\bigcup B(q_i)\right)$$

This theorem along with Lemma 1 proves the Kneser-Poulsen conjecture in dimension 2 (since $2 + 2 = 2 \times 2$).

Connelly and I [BC07b] have shown that Lemma 1 cannot be improved:

Theorem 5 (Belk and Connelly)**.** *There exist configurations* $\mathbf{p}$ *and* $\mathbf{q}$ *in* n*-dimensional Euclidean space such that* $\mathbf{q}$ *is a contraction of* $\mathbf{p}$ *and there is no continuous contraction from* $\mathbf{p}$ *to* $\mathbf{q}$ *in* $2n - 1$ *dimensions.*

In dimension 2, Bezdek and Connelly [BC02] showed that the following contraction cannot be made continuous in dimension 3 (but can be in dimension 4):



The first configuration is a triangle with “flaps” attached to each edge. In the second configuration, the “flaps” have been folded over. It is not hard to see why this contraction requires 4 dimensions to be continuous. Imagine trying to perform the contraction in 3 dimensions — each “flap” would need to move up or down, and adjacent “flaps” would need to move in opposite directions. Since there are an odd number of “flaps”, this cannot be done in 3 dimensions.

In higher dimensions, the configurations $\mathbf{p}$ and $\mathbf{q}$ of Theorem 5 are constructed by attaching “flaps” to the facets of a simplex, similar to how the above configurations involve a triangle with “flaps” attached to the edges. Connelly and I [BC07b] showed that the related bar framework is rigid in dimension $2n-1$, proving that $2n$ dimensions are required for a continuous contraction.

Future Plans: There are many open questions related to the Kneser-Poulsen conjecture with the main question, of course, being whether the conjecture holds in dimensions greater than 2. Here are two directions that I think could be interesting:

1. It could be interesting to use a computer to look for counterexamples to the Kneser-Poulsen conjecture. In particular, it would be interesting to see if the configuration in Theorem 5 might provide a counterexample.

2. The conjecture is also open in spherical space and hyperbolic space. Csikós [Cs02] has proven an analogue of Theorem 4 in these spaces, but it is unknown what dimension is needed to extend every contraction to a continuous contraction. In hyperbolic space, it is not even known if every contraction extends to a continuous contraction.

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