Cops and Robbers

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
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Abstract

"Cops and Robbers" is a mathematical game in which pursuers attempt to capture an evader in a certain environment. In the version of the game that I am studying, the cops and the robber move simultaneously on a connected graph. The robber has the advantage of being infinitely fast, and both the robber and the cops know the other side's location. My goal in this project is to characterize all graphs where three cops are necessary and sufficient to capture the robber. One way to characterize these graphs is to find their forbidden minors, or equivalently, minimal graphs where four cops are needed to capture the robber. I have thus far found seven forbidden minors, and I conjecture that these are the only ones for cop number three.

Contents

List of Figures

Dedication

To my family, with love.

Acknowledgments

I would like to thank my advisor Maria Belk for the considerable amount of time and effort she has spent working with me on this project. She has been endlessly patient and unfailingly supportive, giving me excellent advice and strongly needed criticism. Without her this project would not have been completed. I would also like to thank my two other board members, Sam Hsiao and Melvin Chen, for their suggestions. Finally, I would like to thank Lauren Rose and Robert Martin for being so understanding and giving me so much support with the difficulties that I have faced over senior year.

1

Introduction

"Cops and Robbers" is a mathematical game in which pursuers attempt to capture an evader in a certain environment. In some variants of the game, each side moves in alternate turns, with a certain number of moves per turn. In other variants, the pursuers, or cops, may or may not know the robber's location at every moment. No matter what differences there are between the many versions of the game, the final goal of "Cops and Robbers" is the same for each and every one of these, and that is to find the number of pursuers needed in order to capture the evader.

In the version of the game that I am studying, the cops and the robber move simultaneously, with the robber having the advantage of being infinitely fast, and with the cops always aware of the robber's location at every moment. Connected graphs, with edges and vertices, are the sole environments that the cops and the robber of my game will move in. In the very next chapter, I will give a detailed list of the game's rules as well as some examples of graphs where only three cops are needed in order to capture a robber. Since graphs with cop numbers one and two have already been found, three is therefore the first interesting cop number, and also the one I will study.

The goal of my project is to find a way to characterize all graphs where three cops are necessary and sufficient to capture the robber. Further along in Chapter 2 are definitions and theorems from outside sources that I will be using in my project.

One way of characterizing my graphs with cop number three is to find forbidden minors for them. Chapter 3 proves that certain graphs (seven of them, to be exact) are forbidden minors to all graphs with cop number three. These forbidden minors are simply the minimal graphs where four cops are needed to capture the robber. The chapter ends with a conjecture that these are the only forbidden minors for cop number three.

I have included any other results found from my work on this project in Chapter 4, and I invite the readers to solve a few open questions in Chapter 5.

2
Preliminaries

2.1 Explanation of the Rules of the Game

My research involves the study of the mathematical game “Cops and Robber” which can be pictured as follows: by moving along only on the edges and vertices of a graph, cops try to capture an infinitely fast robber. To be more precise, the cops and the robbers can move at the same time, and not on turns. We will be considering only connected graphs where loops and double edges are allowed. The goal of the game is to find the minimum number of cops needed to capture a robber for a certain graph, which from now on we will be call the cop number.

I will begin by listing the rules of the game “Cops and Robber” in more detail.

Rules of the game:

1. The cops and the robber can only move along the edges and vertices of a graph.
2. The robber can travel at any finite speed.

3. The game ends if the robber is in the same location as a cop. Also, a robber cannot move "through" a cop.

4. The cops and the robber always know the location of all players at all times.

5. The *cop number* $c(G)$ of a graph is the minimum number of cops necessary to catch a robber, with everyone trying out their best strategy.

6. Cops place themselves first. The robber then chooses where to start in a place in which he or she will not be immediately captured.

Let G be a graph. The goal is to find the cop number of G, which we will denote by $c(G)$. We can easily see that only a single cop is needed to capture the robber on a tree while exactly two are needed on a cycle.

As graphs become more complex, it becomes increasingly difficult to find $c(G)$. One way to prove that G has $c(G) = m$ for some integer m is to first show that there is an escape strategy for the robber when there are only $m - 1$ cops, and then show that m cops can capture the robber.

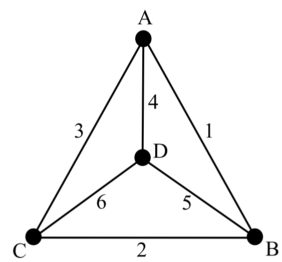
2.2 Some Cop Numbers

The following lemmas are examples of a few basic cop numbers.

Lemma 2.2.1. *Let G be the graph K_4 as shown in Figure 2.2.1. Then* $c(G) = 3$

Proof. Robber can never be captured by 2 cops: Let v be a vertex in G. The degree of all vertices in G is 3, so the degree of v is 3. Since there are 3 other vertices incident to the edges leaving v, then there are 3 paths each leading to three different vertice that the robber can go to in order to escape capture. If a cop goes towards v, the other cop can only occupy one of the three edges incident to the three vertices leading out of v, and therefore the 3rd path—edge and vertex—is always free, which means that as long as the

Figure 2.2.1. This is the graph K_4.



robber keeps to one of the two free vertices or the edges incident to the free vertices, it can escape from the cops indefinitely.

Robber can always be captured by 3 cops: Have 3 cops occcupy vertices A, B, D. If the robber is on edge 1, 4, or 5, both cops on vertices incident to the edge the robber is on converge towards the robber and the robber is captured. If the robber is on vertex C or edge 2, 3, or 6, all 3 cops should converge towards C, then the robber is caught. □

Lemma 2.2.2. *Let G be the graph as shown in Figure 2.2.2. Then $c(G) = 3$.*

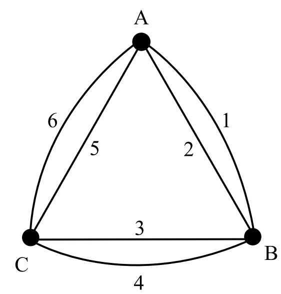
Proof. **Robber can never be captured by 2 cops** Have 2 cops occupy vertices A and B. By symmetry, all the vertices are equivalent in this graph. If the robber is on edge 1 or 2, then the two cops can converge on the the robber and the robber is caught. Therefore, have the robber start on C or an edge incident to C.

One strategy is to have the robber run towards an unoccupied vertex as soon as a cop is halfway on an edge from the vertex it left. There is always a path unblocked by cops to this newly freed vertex since there are only 2 cops, and 2 cops can only block 2 vertices, or 1 vertex and 1 edge, or 2 edges.

Cops block 2 vertices: The robber is left to occupy 3rd vertex.

Cops block 1 vertex and 1 edge: There are two free vertices that the robber can occupy,

Figure 2.2.2. This a triangle with all the edges doubled



so therefore the robber can always escape to another vertex. Since there are four edges incident to every vertex and one cop is still on a vertex, then the robber cannot go on the two of these edges because of this cop. If the third cop is on one of the two edges leading to the free vertex, there is always one other edge that the robber can use to escape from the cops.

Cops block 2 edges: All 3 vertices are free of cops. Since there are four edges incident to the vertex that the robber is on, the cops can only block two of these edges, and therefore the robber can use the other two edges in order to reach a free vertex.

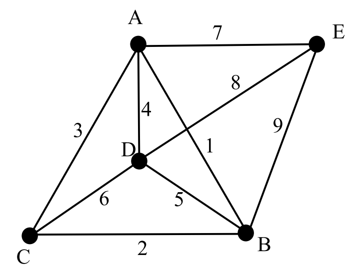
Using this strategy, the robber will never be caught by only 2 cops.

Robber can always be captured by 3 cops: Have each cop occupy a different vertex, so with 3 cops every vertex is occupied. The robber is then forced to place itself on an edge where both ends are occupied by cops. Have the two cops on the vertices incident to the edge that the robber is on converge towards the robber. Then the robber is caught. □

Lemma 2.2.3. *Let G be the graph as shown in Figure 2.2.3. Then* $c(G) = 3$

Proof. Robber can never be captured by 2 cops: Since the robber can survive 2 cops on a subgraph of G (see Figure 2.2.1), then the robber must also be able to survive 2 cops in G without needing to go on edges and vertices in G and not in Figure 2.2.1.

Figure 2.2.3. This is a tetrahedron with another tetrahedron attached to one of its faces



Robber can always be captured by 3 cops: Have the cops start on vertices A, B, and D. The robber can only be on C, E, or an edge incident to C or E, but will be caught when all 3 cops converge towards the robber.

A bad placement for the cops would be C, D, E. Then there is actually a path between the 2 free vertices A and B, so the robber can escape indefinitely using this route. □

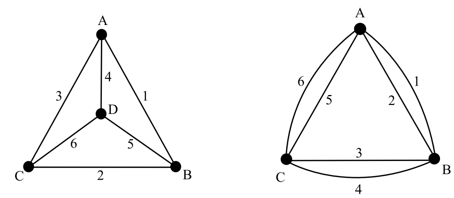
2.3 Motivation

Related to my project is a paper by Seymour and Thomas [1] that deals with a similar cops-and-robber game. The main difference between my game and theirs is that the cops in their game do not travel on the edges. Instead, they are "picked up" and "dropped off" on any vertex they choose by helicopters. In my project, the Seymor and Thomas cop number will be denoted $k(G)$ as opposed to my own cop number which will remain $c(G)$.

In their paper, Seymour and Thomas prove that $k(G)$ is related to the treewidth of a graph according to the following theorem:

Theorem 2.3.1 (Seymour and Thomas)**.** *Let G be a graph. Then $k(G) = tw(G) + 1$.*

Figure 2.3.1. The forbidden minors of graphs with cop number 2



From personal communications with James and Maria Belk, I have learned that the cop number of my own game is related to the treewidth in the following way:

Theorem 2.3.2 (Belk and Belk)**.** *Let G be a graph. Then* $tw(G) \leq c(G) \leq tw(G) + 1$.

In an article by Arnborg [3], all graphs with treewidth at most 3 have a finite list of forbidden minors that characterize the graph. Therefore, we will attempt to find the finite list of forbidden minors that characterize all graphs with $c(G) = 3$. A further source of motivation comes from a personal communication from Maria Belk that the forbidden minors for our game with cop number 2 are K_4 and the double-edged triangle as shown in Figure 2.3.1.

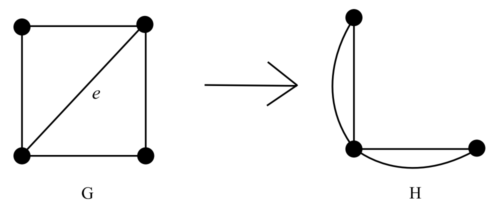
2.4 Background

This section will include the theorems and definitions that are needed for my project. First, we will define what a minor is. The following definition is a standard definition from graph theory [4].

Definition 2.4.1. A graph H is a *minor* of G if H is isomorphic to a graph obtained from G by contracting edges, deleting edges, and deleting isolated nodes. △

Example 2.4.2. By contracting the edge e from graph G, we get H, a minor of G as shown in Figure 2.4.1

Figure 2.4.1. Contracting edge e from graph G gives H, a minor of G



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The following theorem comes from personal communication with James and Maria Belk and relates minors to the cop numbers of my game:

Theorem 2.4.3 (Belk and Belk)**.** *If H is a minor of G then $c(H) \leq c(G)$*

The following theorem comes from an article by Robertson and Seymour [5].

Theorem 2.4.4 (Graph Minor Theorem)**.** *For every family $\mathcal{F}$ of graphs such that if a graph is in the family then all its minors also are, there is a finite class O of graphs such that a graph G is in $\mathcal{F}$ if and only if no member of O is a minor of G. The members of O are called the forbidden minors for the family $\mathcal{F}$.*

1. If no member of O is a minor of a graph G in $\mathcal{F}$, then G is in $\mathcal{F}$.

2. If a graph G is in $\mathcal{F}$, then no member of O is a minor of G.

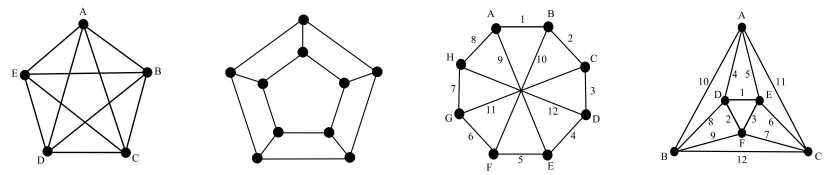
The following theorem is from an article by S. Arnborg et al. [3]

Theorem 2.4.5. *The forbidden minors of graphs with treewidth less than or equal to 3 are shown in Figure 2.4.2.*

For the following definitions, we will need to define a complete graph K_n. Let K_n be the graph with n pairwise adjacent vertices without any multiple edges or loops.

From an article by Belk and Connelly [2], we will use the following three definitions:

Figure 2.4.2. The forbidden minors of graphs with treewidth less than or equal to 3



Definition 2.4.6. Let G_1 and G_2 be two graphs, both containing a K_k as a subgraph. The *k-sum* of G_1 and G_2 denoted $G_1 \oplus_k G_2$ is the graph obtained by identifying the two K_k's. △

Definition 2.4.7. A graph is a *k-tree* if it can be obtained through a sequence of *k-sums* of K_{k+1}'s. A graph is a partial *k-tree* if it is a subgraph of a *k-tree.* △

The following graphs as shown in Figure 2.4.3 are examples of 2-trees, 3-trees, and 4-trees. A 1-tree is simply a normal tree.

Definition 2.4.8. The *treewidth* of G is the minimum k such that G is a subgraph of a k-tree. △

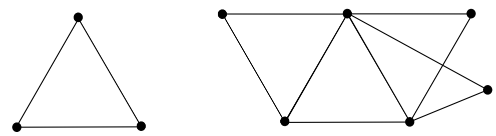
The following is another definition of treewidth [4]:

Definition 2.4.9. Let G be a graph and let $v_1, ..., v_n$ be vertices in G. The *treewidth* of G is less than or equal to k if there exists a tree T with vertices $T_1, ..., T_m$ such that:

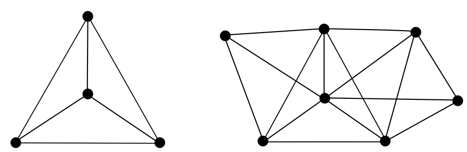
1. Each T_i is a set containing $\leq k+1$ vertices of G.
2. $\bigcup T_i =$ all vertices of G.
3. If u, v is an edge in the graph G, then there is a T_i that contains both u and v.
4. If $T_i \bigcap T_j \neq \emptyset$ and T_k is a vertex on the path from T_i to T_j then $T_i \bigcap T_j \subseteq T_k$.

△

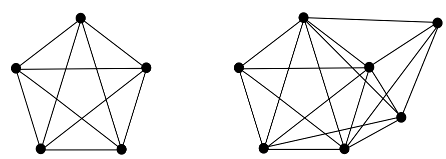
Figure 2.4.3. Examples of k-trees



2-trees

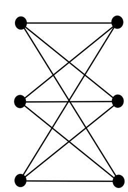


3-trees



4-trees

Figure 2.5.1. $K_{3,3}$



2.5 Example of Minor and Treewidth of $K_{3,3}$

Example 2.5.1. To clarify this topic, let us find a minor of $K_{3,3}$ and determine its treewidth, as shown in Figure 2.5.1

By contracting and deleting the edges as shown in Figure 2.5.2, $K_{3,3}$ has K_4 as a minor and since K_4 is a 3-tree, the treewidth of $K_{3,3}$ cannot be smaller than 3.

To prove that the treewidth of $K_{3,3}$ is 3, we must show that $K_{3,3}$ is a subgraph of a 3-tree. Add an edge from B to D, an dedge from D to F, and a final edge from B to

Figure 2.5.2. By contracting and deleting the following edges, $K_{3,3}$ has K_4 as a minor:

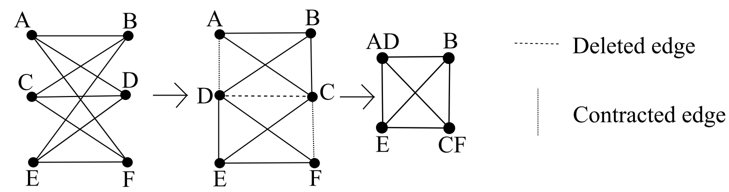
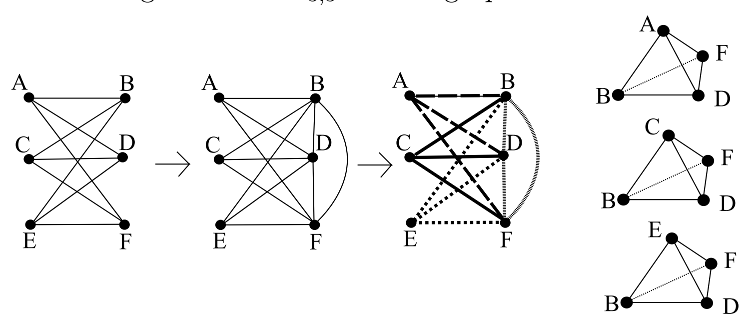


Figure 2.5.3. $K_{3,3}$ is a subgraph of a 3-tree.



F to the $K_{3,3}$ depicted in Figure 2.5.3. Then the resulting graph is a 3-tree, with three tetrahedrons sharing the same face formed by B, D, and F.

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2.6 Results of the Project

As of this moment, I have found two theorems. Chapter 3 covers the proof of the first theorem, while the second theorem is proved later in Chapter 4. The theorems are the following:

Theorem 2.6.1. *If G has $c(G) \leq 3$, then G does not have $FM_1, FM_2, ..., FM_7$ as a minor.*

Theorem 2.6.2. *If G satisfies the following conditions, then $c(G) \leq 3$.*

1. $tw(G) \leq 3$

2. *There exists a tree satisfying all the clauses from Def 2.4.9 and such that if* $T_a \subseteq V(T)$ *and* $T_1, T_2, ..., T_m$ *are neighbours of* T_a*, then* $\mid T_1 \bigcap T_2 \bigcap ..., \bigcap T_m \mid \geq 2$.

I conjecture that the converse holds for both these theorems.

3
Forbidden Minors

As of now, I have found seven different forbidden minors for $c(G) = 3$ (see Figure 3.0.1).

To prove that these seven graphs are forbidden minors for cop number 3, we need to show the following for each of the graphs:

1. Show that the robber can never be captured by 3 cops.
2. Show that the robber can always be captured by 4 cops.
3. Show that the deletion or the contraction of any edge of the graph results in a graph with cop number 3. Therefore the graph is a minimal graph with cop number 4.

3.1 FM_1

Lemma 3.1.1. *Let* FM_1 *be the graph as shown in Figure 3.1.1. Then* $c(FM_1) = 4$ *and* FM_1 *is a forbidden minor for the game with 3 cops.*

Proof. Robber can never be captured by 2 cops: The robber cannot be captured by 2 cops because it survives 2 cops on one of its subgraphs (see Figure 2.2.3).

Figure 3.0.1. The seven forbidden minors for $c(G) = 3$.

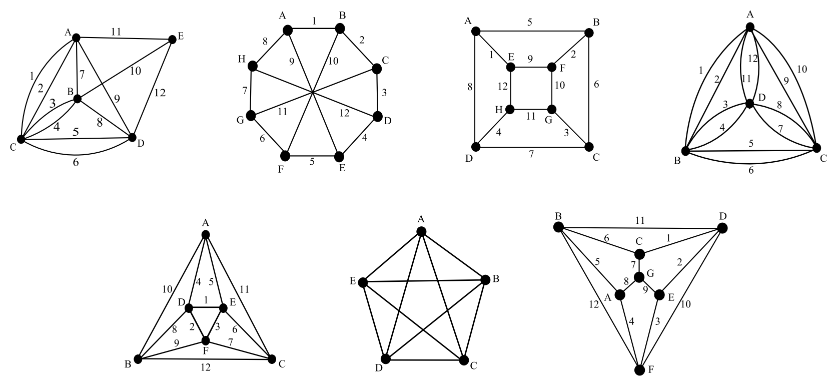
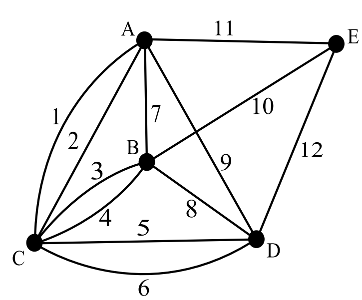


Figure 3.1.1. FM_1



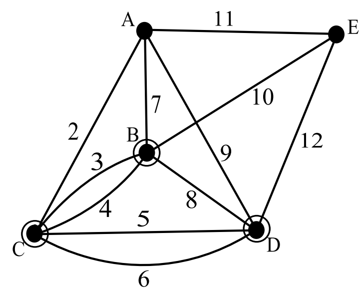
Robber can never be captured by 3 cops: Have robber start on any vertex except E (in which case cops can converge from vertices A, B, D).

By symmetry, vertices A, B and D are equivalent, and there are roughly three families of equivalent edges. Let Family 1 be edges $1, 2, 3, 4, 5, 6$, Family 2 edges $7, 8, 9$, and Family 3 edges $10, 11, 12$. Now, suppose the robber choses to start on vertex C. Then the preferable strategy for the cops is to occupy the vertices A, B, and D. When a cop is halfway on an edge incident to C, the cop frees up the vertex it just occupied, so have the robber occupy this freed-up vertex. Now the robber is on one of the vertices A, B, or D. If the cop now on C starts to advance towards the robber, have the robber return to occupy C. If the cop on C remains on C, and the other 2 cops converge at the same time upon the robber on the two edges from 7, 8, or 9 that are incident to the vertex the robber occupies, then the robber can escape using the edge leading to E. Once on E, make sure robber goes to the one unoccupied vertex among A, B, or D as soon as the cop on C moves, because if all 3 cops are on A, B, or D, then robber on E (or one of the edges incident to E) is caught. If the robber is back on A, B, or D, then repeat the steps mentioned earlier. By this strategy, the robber can escape from 3 cops indefinitely.

Robber can always be captured by 4 cops: Have the cops start on A, B, C, and D. The robber is caught when at least 3 cops converge towards E if it picks E or any of the edges incident to E. So have robber choose an edge not incident to E. If the robber picks some edge not incident to E, then there are two cops on the vertices on both ends of any edge the robber can be on. Have the cops converge on the robber from both ends of the edge, then the robber is caught.

FM_1 **is a minimal graph with** $c(FM_1) = 4$**:** By symmetry, there are three families of equivalent edges, and we will only need to show that the graphs obtained by contract-

Figure 3.1.2. This is FM_1 with edge 1 deleted



ing or deleting any one edge from each of these three families all have cop number 3. The circled vertices represent the starting locations of the cops.

Family 1: Edges $1, 2, 3, 4, 5,$ **and** 6:

Edge 1 deleted. See Figure 3.1.2:

Robber can always be captured by 3 cops: Have the cops start on C, B, and D. Then the robber can start either on vertices A or E. If the robber starts on vertex A, then have the cop on C move towards A. The robber will only be able to retreat towards E. Have all three cops converge (from A, B, and D) towards E. The robber is caught.

Edge 1 contracted. See Figure 3.1.3:

Robber can always be captured by 3 cops: Have the cops start on the three vertices AC, B, and D. If the robber starts on vertex E or any edge incident to E, the three cops can converge on E and robber is captured. The robber cannot be on edges 3 to 9 because there are cops on both ends of any edge 3-9, and therefore robber is caught. If the robber starts on edge 2, which is a loop, have one cop stay on vertex AC and have another cop move into the loop in either a clockwise or counter-clockwise direction. The robber is caught when the second cop has finished moving throughout the loop and is back on AC.

Figure 3.1.3. This is FM_1 with edge 1 contracted

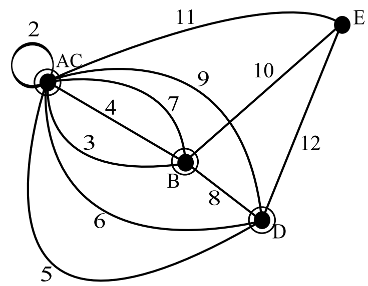
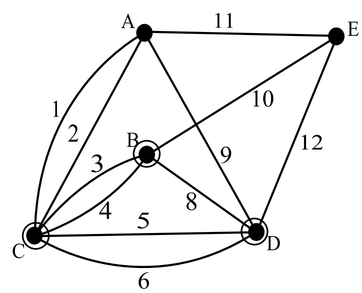


Figure 3.1.4. This is FM_1 with edge 7 deleted



Family 2: Edges $7, 8,$ **and** 9

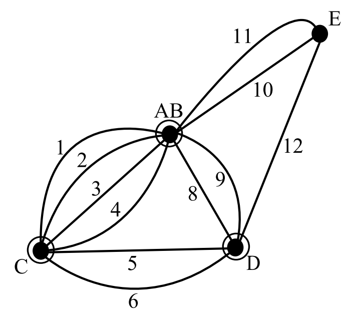
Edge 7 deleted. See Figure 3.1.4:

Robber can always be captured by 3 cops: Have the cops start on B, C, and D. Then the robber can start on A or E. If the robber starts on A, have the cop on B move towards E. Then have cops on E and D converge towards A. The robber will have retreated to edge 1 or 2, where it is caught once the cops converge upon it from both ends of the edge. If the robber started on E, have the cop on B move towards E, then do the same steps as before. Robber can always be captured on this graph by 3 cops.

Edge 7 contracted. See Figure 3.1.5:

Robber can always be captured by 3 cops: Have the cops start on vertices AB, C, and D. Then the robber can only start on E or any of the edges incident to E. Have the cop on C

Figure 3.1.5. This is FM_1 with edge 7 contracted



move to AB. Now, have one cop on AB take edge 11 while the other cop on AB moves on edge 10 towards E while third cop stays on D. The robber is caught on edge 12 between cops that occupy vertices on both sides of the edge, so the robber is caught.

Family 3: Edges $10, 11,$ **and** 12

Edge 10 deleted. See Figure 3.1.6:

Robber can always be captured by 3 cops: Have the cops start on vertices A, C, and D. Now, if the robber starts on E or any edge incident to E, have the cops on A and D converge towards E and the robber is caught. If the robber starts on B or any edge incident to B, have the cop on A move on edge 7 and cop on D move on path 8 towards B while cop on C stays still. Then the robber is forced into edges 3 or 4, and therefore the robber is caught.

Edge 10 contracted. See Figure 3.1.7:

Robber can always be captured by 3 cops: Have the cops start on vertices A, C, and BE. Now the robber can only go on D or edges incident to D. Have the cop on A move on edge 9 towards the robber on D. The robber is caught in edges 5, 6, 8, or 12. Since there are cops on all vertices on both ends of each of these edges, the robber is caught.

□

Figure 3.1.6. This is FM_1 with edge 10 deleted

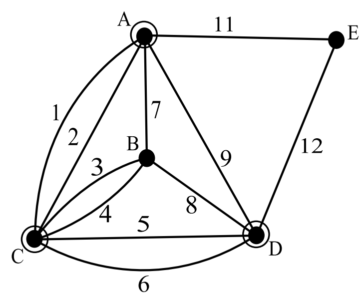


Figure 3.1.7. This is FM_1 with edge 10 contracted

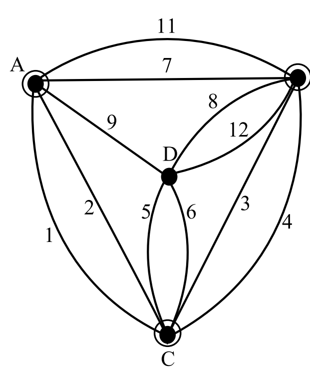
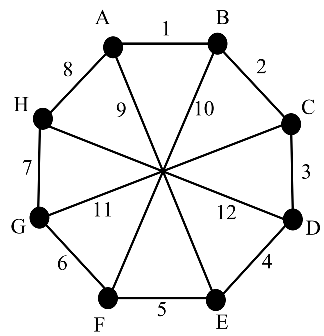


Figure 3.2.1. FM_2



3.2 FM_2

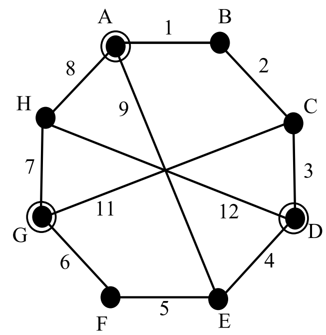
Lemma 3.2.1. *Let FM_2 be the graph as shown in Figure 3.2. Then $c(FM_2) = 4$ and FM_2 is a forbidden minor for the game with 3 cops.*

Proof. Robber can never be captured by 3 cops: By Theorem 2.4.5, FM_2 is one of the forbidden minors for treewidth 3. Therefore $tw(FM_2) = 4$. Now from Theorem 2.3.2, $4 \leq tw(FM_5) \leq c(FM_5)$ and therefore at least 4 cops are needed in order to capture the robber, which means that the robber can always escape from 3 cops.

Robber can always be captured by 4 cops: Have the cops start on vertices B, H, E and D. Then the robber cannot start on A and can only start on C, G, or F. Have the cop on H move to G. If the robber is on C or any of the edges incident to C (edges 2, 3, or 11), then cops on B, G, and D converge on robber and robber is caught. If robber is on F or edges incident to F, then cops on G, H, and B converge on robber and robber is caught. There are no other possible places for the robber to be on, so the robber can always be caught by 4 cops.

FM_2 **is a minimal graph with** $c(FM_2) = 4$**:** By symmetry, there are two families of

Figure 3.2.2. FM_2 with one of the inside edges deleted



equivalent edges: the inside edges $(9, 10, 11,$ or $12)$ and the outside edges $(1, 2, 3, 4, 5, 6, 7,$ or $8)$. We need to show that the graphs obtained by contracting or deleting any one edge from each of these three families all have cop number 3. The circled vertices represent the starting locations of the cops.

Inside Edges: Edges $9, 10, 11,$ or 12.

Edge 10 deleted. See Figure 3.2.2:

Robber can always be captured by 3 cops: Have the cops start on $A, G,$ and D. Then the robber cannot start on H. If robber starts on B or C, have cop on A go to B while other two cops remain on G and D. The robber is now forced to be on B. Then have all three cops from $B, G,$ and D converge on robber on C. By symmetry, it is equivalent if robber starts on F or E instead of B or C. Have the cop on G go to F and now have three cops on $A, F,$ and D converge towards . The robber is caught.

Edge 10 contracted. See Figure 3.2.3:

Robber can always be captured by 3 cops: Have the cops start on $H, F,$ and D. By symmetry, A is equivalent to C, G is equivalent to E, and D is equivalent to H. If the robber starts on A, have cop on D go to E, now have all three cops on $H, F,$ and C converge onto robber on A. If the robber starts on G, have cop on D go to C, and now have all three cops on $H, F,$ and C converge on robber on G.

Figure 3.2.3. FM_2 with one of the inside edges contracted

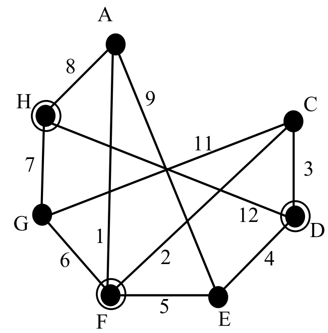
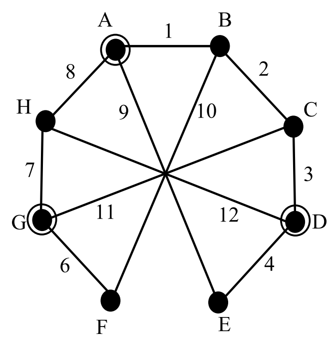


Figure 3.2.4. FM_2 with one of the outside edges deleted

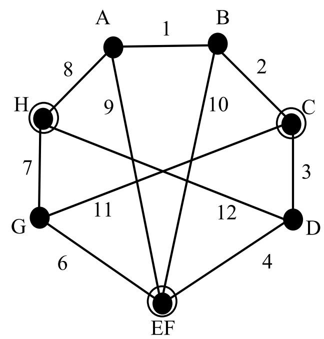


Outside Edges: Edges $1, 2, 3, 4, 5, 6, 7$, or 8.

Edge 5 deleted. See Figure 3.2.4:

Robber can always be captured by 3 cops: Have the cops start on D, A, and G. Then the robber cannot start on E or the two cops on D and A can converge on E and the robber is captured. The robber also cannot start on H or the cops on D, A, and G converge towards H and the robber is captured. The robber can start on B, C or F. If the robber starts on B or C or F or any edges incident to these vertices, have the cop on D go to C while the other two cops stay on A and G. Now the robber can only be on B or F or any of the edges incident to B or F. Have the cop on G go to F, and now have all three cops from A, F, and C converge towards B. Then the robber is caught.

Figure 3.2.5. FM_2 with one of the outside edges contracted



Edge 5 contracted. See Figure 3.2.5:

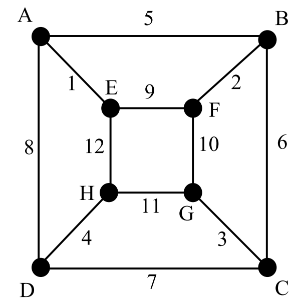
Robber can always be captured by 3 cops: Have the cops start on EF, H and C. The robber cannot start on G or D because the three cops can converge on either one of these vertices and the robber would be captured. The robber can start on either A or B. By symmetry, we will only do one of these. Suppose the robber starts on A. Then, have the cop on C move towards B. Now, have all three cops on H, EF, and B converge upon A and the robber is caught. □

3.3 FM_3

Lemma 3.3.1. *Let FM_3 be the graph as shown in Figure 3.3.1. Then $c(FM_3) = 4$ and FM_3 is a forbidden minor for the game with 3 cops.*

Proof. Robber can never be captured by 3 cops: When there are 2 or 3 cops on the inside cycle (vertices E, F, G, H and edges 9, 10, 11, 12), have the robber stay on the outside cycle (vertices A, B, C, D and edges 5, 6, 7, 8). When 2 or 3 cops are on the outside cycle, the robber should stay on the inside cycle. When a cop is travelling from the inside cycle to the outside one, it must use one of the 4 middle edges (1, 2, 3, 4). The robber should take the a free middle edge to go onto the cycle that the cop just left. There is

Figure 3.3.1. FM_3



always one edge free because there are only 3 cops and there are 4 possible edges to use.

Robber can always be captured by 4 cops: Have the cops start on vertices B, E, G, and D. Then if the robber starts on A or edges incident to A, three cops from D, E, and B would converge on A and the robber would be caught. If the robber starts on H or edges incident to H, three cops from D, E, and G would converge on H and the robber would be caught. If the robber starts on F or edges incident to F, three cops from B, E, and G would converge on F and the robber would be caught. Finally, if the robber starts on C or edges incident to C, three cops from D, B, and G would converge on C and the robber would be caught.

FM_3 **is a minimal graph with** $c(FM_3) = 4$**:** By symmetry, all edges are equivalent. We need to show that the graphs obtained by contracting or deleting any one edge have cop number 3. The circled vertices represent the starting locations of the cops.

Edge 7 deleted. See Figure 3.3.2:

Robber can always be captured by 3 cops: Have the cops start on vertices A, H, and F. If the robber starts on E or any edge incident to E, have the cops converge upon E on all the same time and robber is caught. If the robber starts on D or edges incident to D,

Figure 3.3.2. FM_3 with an edge deleted

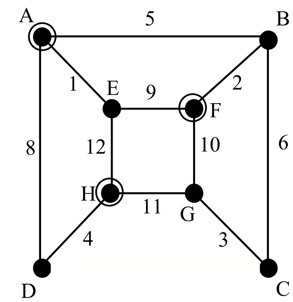
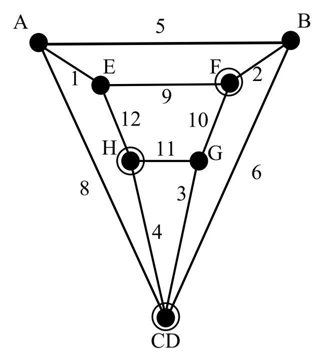


Figure 3.3.3. FM_3 with an edge contracted

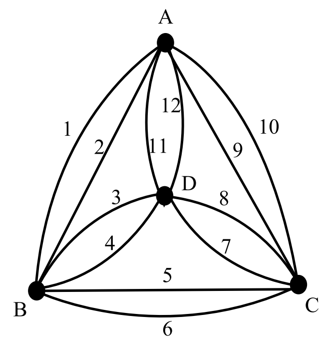


have cops on A and H converge upon D and the robber is caught. If the robber starts on B, G, or C or any of the edges incident to these three vertices, have cop on A go to B using edge 5 and then to C using edge 6 while the other two cops stay on F and H. Now the robber must be on G or any of the edges incident to G. Have the cops on F, H, and C converge upon G. Then the robber is caught.

Edge 7 contracted. See Figure 3.3.3:

Robber can always be captured by 3 cops: Have the cops start on vertices CD, H, and F. Tthe robber cannot start on G or any edges incident to G. Have the robber start on A, E, or B. Now, have the cop on H move to E using edge 12 while other two cops stay on CD and F. Have cop on E go to A using edge 1. If robber is on edge 8, then cops on A and CD converge on robber and robber is captured. If robber is on B or any edge incident to

Figure 3.4.1. FM_4



B, have the three cops on A, F and CD converge towards B and therefore the robber is caught.

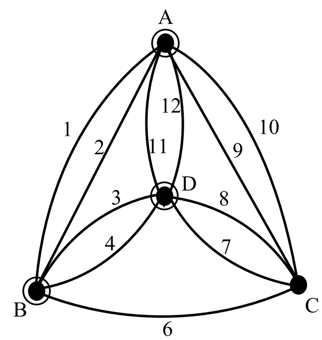
□

3.4 FM_4

Lemma 3.4.1. *Let FM_4 be the graph as shown in Figure 3.4. Then $c(FM_4) = 4$ and FM_4 is a forbidden minor for the game with 3 cops.*

Proof. Robber can never be captured by 3 cops: By symmetry, the vertices are all equivalent. If the cops start on vertices A, B and C, then the robber must start on D or any of the 6 edges incident to D. As soon as a cop is halfway on one of the edges between its own vertex and D, have the robber go to the vertex that the cop just left. The placement of the other two cops does not affect this escape strategy. If the other two cops both occupy vertices, then a third vertex is free for the robber to occupy and only one edge out of two leading to that vertex can be taken by a cop, leaving the other edge free for the robber to use. If the other two cops are on an edge and a vertex, then there are two free vertices and a path must always be available to them because there are not enough cops on the edges to block the cop from leaving its own vertex. If the other two

Figure 3.4.2. FM_4 with an edge deleted



cops are both on edges, then all three cops are on edges and all four vertices are free for the robber to occupy. There must be available paths for the robber to take because if the cops were only on edges, a minimum of six cops would be needed to prevent the robber from leaving D.

Robber can always be captured by 4 cops: Have the cops start on A, B, C, and D. The robber can only start on an edge between cops that occupy vertices on both ends of the edge. Since the cops know which edge the robber is on, two cops on each end of the edge can converge upon the robber and the robber is caught.

FM_4 **is a minimal graph with** $c(FM_4) = 4$**:** By symmetry, all the edges are equivalent. We need to show that the graphs obtained by contracting or deleting any one edge have cop number 3. The circled vertices represent the starting locations of the cops.

<u>Edge 5 deleted. See Figure 3.4.2</u>:

Robber can always be captured by 3 cops: Have the cops start on A, B, and D. Then the robber must start on C or an edge incident to C. Have the cop on B go to C using edge 6. Then the robber must be on edge 7, 8, 9, or 10. Have the two cops on both ends of the edge that the robber is on converge upon the robber. The robber is captured.

Figure 3.4.3. FM_4 with an edge contracted

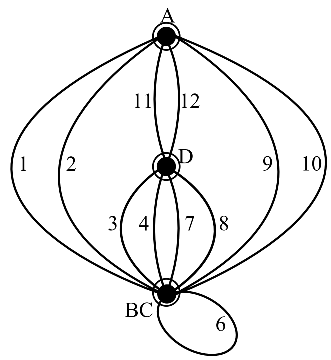
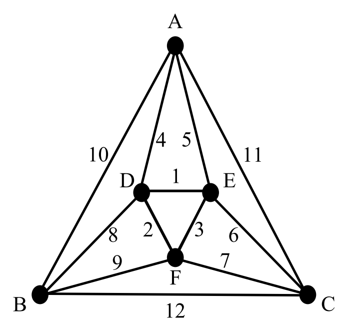


Figure 3.5.1. FM_5



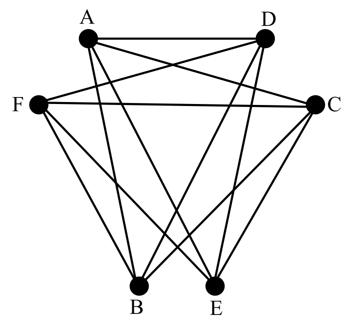
<u>Edge 5 contracted. See Figure 3.4.3:</u>

Robber can always be captured by 3 cops: Have the cops start on A, BC, and D. If the robber is on any edge except for the loop (edge 6), have the two cops on both ends of any edge the robber is on converge on the robber and the robber is captured. If the robber is on the loop, have the cop on BC stay on BC while another cop starts on BC and moves through the entire loop back to BC. The robber is caught. □

3.5 FM_5

Lemma 3.5.1. *Let FM_5 be the graph as shown in Figure 3.5. Then $c(FM_5) = 4$ and FM_5 is a forbidden minor for the game with 3 cops.*

Figure 3.5.2. FM_5 is isomorphic to the graph $K_{2,2,2}$



Proof. Robber can never be captured by 3 cops: By Theorem 2.4.5, FM_5 is one of the forbidden minors for treewidth 3. Therefore $tw(FM_5) = 4$. Now from Theorem 2.3.2, $4 \leq tw(FM_5) \leq c(FM_5)$ and therefore at least 4 cops are needed in order to capture the robber, which means that the robber can always escape from 3 cops.

Robber can always be captured by 4 cops: Have the cops start on B, C, D and E. If the robber starts on A or any edges incident to A, have the cops converge on A and the robber is caught. If the robber starts on F or any edges incident to F, have the cops converge on F and the robber is caught. If the robber starts on edge 1, 8, 6, or 12, have the cops on the vertices that are on both ends of the edge that the robber is on converge upon the robber. The robber is caught.

FM_5 **is a minimal graph with** $c(FM_5) = 4$**:** FM_5 is the graph $K_{2,2,2}$ as depicted in Figure 3.5.2. Therefore, all the edges are equivalent. We need to show that the graphs obtained by contracting or deleting any one edge have cop number 3. The circled vertices represent the starting locations of the cops.

Edge 6 deleted. See Figure 3.5.3:

Robber can always be captured by 3 cops: Have the cops start on A, B and F. The robber

Figure 3.5.3. FM_5 with an edge deleted

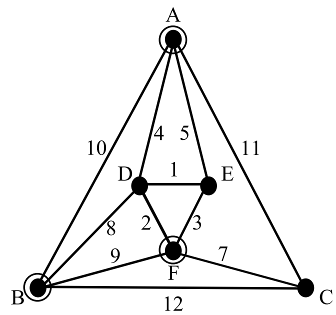
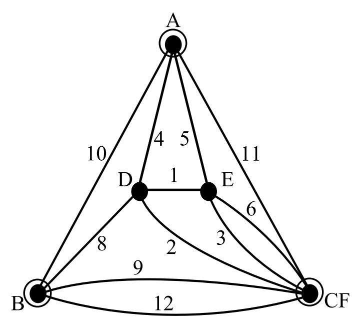


Figure 3.5.4. FM_5 with an edge contracted

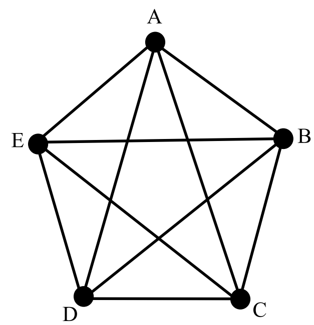


cannot start on C or the three cops on A, B and F can converge on C and the robber is caught. Have the robber start on D or E. Have the cop on B move to D using edge 8. Now the robber can only be on vertex E or the edges incident to E. Have all three cops on A, D and F converge towards E. The robber is caught.

Edge 7 contracted. See Figure 3.5.4:

Robber can always be captured by 3 cops: Have the cops start on vertices CF, A and B. Then the robber can only start on D, E, or any of the edges incident to D and E. Have the cop on B move to D using edge 8. If the robber is on edge 2 or 4, then the two cops on both ends of these edges should converge on the robber and the robber is caught. If the robber is on E or any of the edges incident to E, have the cops on A and D converge

Figure 3.6.1. FM_6



towards E using edge 5 and 1 respectively while the third cop stays on CF. The robber is caught. □

3.6 FM_6

Lemma 3.6.1. *Let FM_6 be the graph as shown in Figure 3.6. Then $c(FM_6) = 4$ and FM_6 is a forbidden minor for the game with 3 cops.*

Proof. Robber can never be captured by 3 cops: By Theorem 2.4.5, FM_6 is one of the forbidden minors for treewidth 3. Therefore $tw(FM_6) = 4$. Now from Theorem 2.3.2, $4 \leq tw(FM_6) \leq c(FM_6)$ and therefore at least 4 cops are needed in order to capture the robber, which means that the robber can always escape from 3 cops.

Robber can always be captured by 4 cops: Have cops start on vertices A, B, C, and D. Then the robber can only be on vertex E or any of the edges incident to E. Have the cops converge on E. Then the robber is caught.

FM_6 **is a minimal graph with** $c(FM_6) = 4$**:** By symmetry, all the edges in FM_6, or more commonly known as K_5 are equivalent. Therefore, we need to show that the two

Figure 3.6.2. FM_6 with an edge deleted

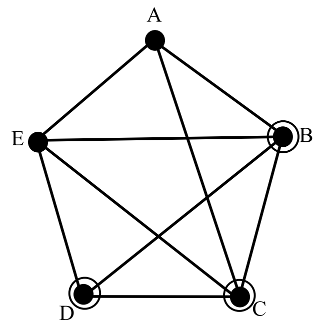
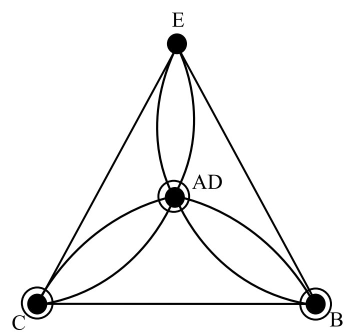


Figure 3.6.3. FM_6 with an edge contracted



graphs obtained by contracting or deleting any one edge have cop number 3. The circled vertices represent the starting locations of the cops.

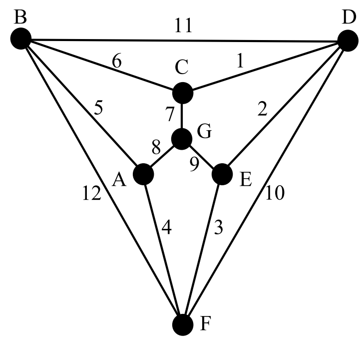
<u>Edge from A to D is deleted. See Figure 3.6.2</u>:

Robber can always be captured by 3 cops: Have the cops start on B, C, and D. Now the robber can only start on A or E or any edges incident to either vertex. Have the cop on D move to E while the other two cops remain on B and C. Now the robber can only be on A or any of the edges incident to A. Have the cops on E, C and B converge towards A. The robber is caught.

<u>Edge from A to D is contracted. See Figure 3.6.3</u>:

Robber can always be captured by 3 cops: Have the cops start on vertices B, C, and AD.

Figure 3.7.1. FM_7



Then the robber must be on E or any of the edges incident to E. Have the cops on B and C both move towards E using the direct edge in both cases. Then the robber will be forced to go to one of the two edges between E and AD. Have two cops on either end of the edge that the robber is on converge towards the robber. The robber is caught. □

3.7 FM_7

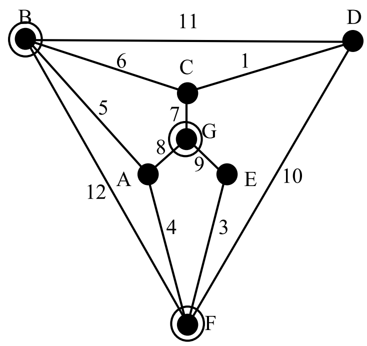
Lemma 3.7.1. *Let FM_7 be the graph as shown in Figure 3.7. Then $c(FM_7) = 4$ and FM_7 is a forbidden minor for the game with 3 cops.*

Proof. Robber can never be captured by 3 cops: The robber is going to attempt to occupy B, D, or F as much as possible. If the cops are on all three of these, have the robber occupy G.

We have two cases depending on where the cops are. The first case is when all three cops are on the outside vertices B, D, and F. The second is when one or more of the cops are not on B, D, or F.

Case 1: If the cops are on B, F and D, the three outside vertices with degree 4, the robber can start on any of the four middle vertices A, C, G and E. Have the robber start on G. If one or more cop(s) leave B, F or D and move towards the robber on G, have the

Figure 3.7.2. FM_7 with one of the edges from Family 1 deleted

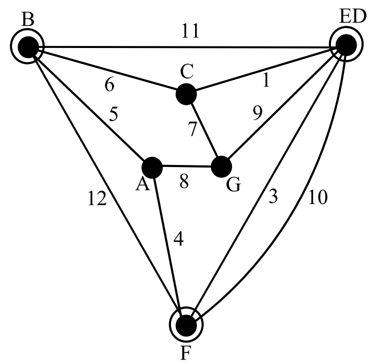


robber leave G as soon as a cop is halfway on an edge incident to B, F or D and have the robber occupy the vertex that the cop has just left. There is always a path that the robber can use because there are 2 disjoint paths from G to any of the outside vertices, and a single cop can only block one of these two paths.

Case 2: If one or more of the cops are not on B, D, or F, then one or more of the outside vertices is free for the robber to occupy. Have the robber start on the free outside vertex. If a cop approaches the robber, and the two other cops are on the outside vertices, have the robber go to G. There must always be a path to G from that vertex because there are two disjoint paths between G and any of the outside vertices. If a cop approaches the robber on an outside vertex, and the other two cops occupy one or less of the outside vertices, have the robber go to another free outside vertex. There is always a path between any two outside vertices when there are only 3 cops because there are 4 disjoint paths between any two outside vertices.

Robber can always be captured by 4 cops: Have the cops start on B, D, F and G. If the robber starts on A, C, or E, then have the three cops that occupy vertices adjacent to the robber converge upon the robber. Then the robber is caught.

Figure 3.7.3. FM_7 with one of the edges from Family 1 contracted



FM_7 **is a minimal graph with** $c(FM_7) = 4$**:** By symmetry, there are three families of equivalent edges: Let Family 1 be edges 1, 2, 3, 4, 5, and 6, Family 2 be edges 7, 8, and 9, and Family 3 be edges 10, 11, and 12. We need to show that the graphs obtained by contracting or deleting any one edge from each of these three families all have cop number 3. The circled vertices represent the starting locations of the cops.

Family 1: Edges 1, 2, 3, 4, 5, and 6.

Edge 2 deleted. See Figure 3.7.2: Robber can always be captured by 3 cops: Have the cops start on B, F, and G. Then the robber cannot start on A or E or the cops can converge on either one of these and the robber is captured. Have the robber start on C or D. Have the cop on F move to D. Then the robber can only be on C or an edge incident to C. Have the cops on B, G, and D converge towards C. The robber is caught.

Edge 2 contracted. See Figure 3.7.3: Robber can always be captured by 3 cops: Have the cops start on B, ED, and F. Have the robber start on C, A, or G. Have the cop on F move to A, and then have the cop on B move to C using edge 6. Then the robber must be on G or any edge incident to G. Have all three cops converge towards G. The robber is caught.

Figure 3.7.4. FM_7 with one of the edges from Family 2 deleted

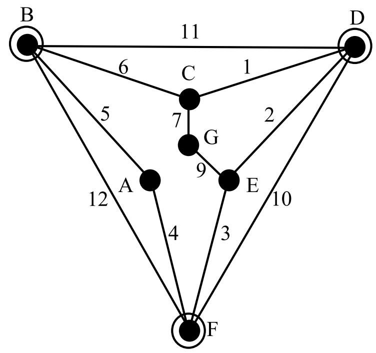
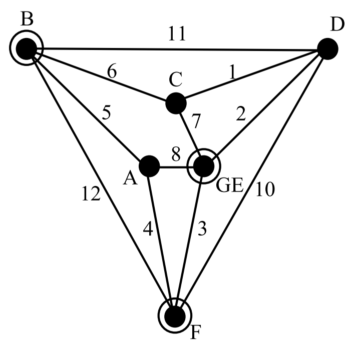


Figure 3.7.5. FM_7 with one of the edges from Family 2 contracted



Family 2: Edges 7, 8, and 9

Edge 8 deleted. See Figure 3.7.4: Robber can always be captured by 3 cops: Have the cops start on B, D and F. Then the robber cannot start on A or the cops on B and F can converge towards A and the robber is captured. Have the robber start on C, G, or E. Have the cop at B move to C and the cop at F move to E. If the robber is on edges 1 or 2, cops on C, E, and D can converge on the robber and capture it. If the robber is on G or an edge incident to G, then have cops on C and E converge on G, and the robber is captured.

Edge 9 contracted. See Figure 3.7.5: Robber can always be captured by 3 cops: Have the cops start on B, F, and GE. Then the robber cannot start on A or the cops can converge on A and the robber is captured. Have the robber start on C or D. Have the cop at F

Figure 3.7.6. FM_7 with one of the edges from Family 3 deleted

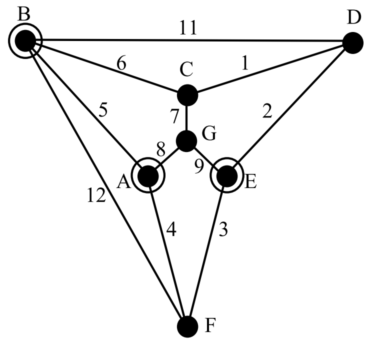
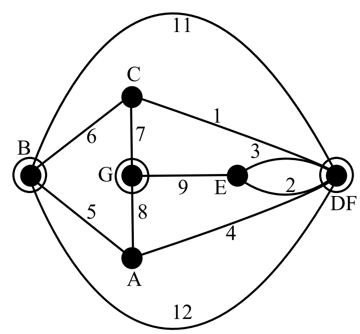


Figure 3.7.7. FM_7 with one of the edges from Family 3 contracted



move to D, then the robber must be on C or any of the edges incident to C. Have the cops on B, D, and GE converge towards C. The robber is caught.

Family 3: Edges 10, 11, and 12.

Edge 10 deleted. See Figure 3.7.6: Robber can always be captured by 3 cops: Have the cops start on A, B and E. Then the robber cannot start on F. Have the robber start on C, G, or D. Have the cop on A move to G, and then to C. Then the robber must be on D or an edge incident to D. Have the three cops on B, C, and E converge towards D. Then the robber is caught.

Edge 10 contracted. See Figure 3.7.7: Robber can always be captured by 3 cops: Have the cops start on B, G and DF. Then the robber cannot start on A or C or the cops can converge towards either vertex and the robber will be caught. Have the robber start on E or an edge incident to E. Have the cop on G move to E. Then the robber is on edge 2 or 3. Have the cops on E and DF converge on the robber. The robber is caught. $\square$

4

Other Results

Lemma 4.0.2. *Let G be a connected graph without loops or multiple edges. Suppose that G contains k vertices such that any two of them have k-disjoint paths between them. Then G has cop number at least k.*

Proof. Robber can never be captured by $k-1$ cops: If the $k-1$ cops start on any $k-1$ vertices, there is one vertex left that the robber can occupy. If one of the cops leaves its vertex to move towards the robber, then one more vertex is freed for the robber to occupy. Since there are k-disjoint paths between these two vertices—the one that the robber is on and the one that the cop left—the $k-1$ cops are not enough to block the robber. Thus it requires at least k cops to capture the robber. □

Corollary 4.0.3. *Suppose that G is k-connected, then G has cop number at least k.*

Proof. By Menger's Theorem from Graph Theory [7], G satisfies the condition of Lemma 4.0.2. Then it follows that G has cop number at least k. □

Figure 4.0.1. 3-tree where a tetrahedron has tetrahedrons on more than 2 faces.

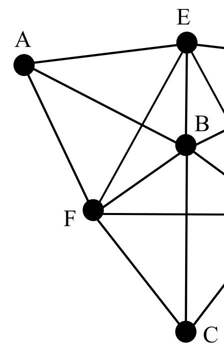
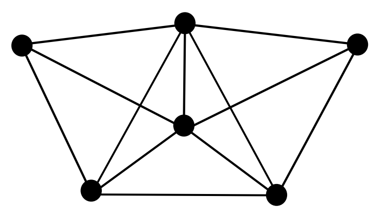


Figure 4.0.2. 3-tree where every tetrahedron has an adjacent tetrahedron on at most 2 faces.



Theorem 4.0.4. *If G satisfies the following conditions, then $c(G) \leq 3$.*

1. *$tw(G) \leq 3$*
2. *There exists a tree satisfying all the clauses from Def 2.4.9 and such that if $T_a \subseteq V(T)$ and $T_1, T_2, ..., T_m$ are neighbours of T_a, then $\mid T_1 \bigcap T_2 \bigcap ..., \bigcap T_m \mid \geq 2$.*

A rephrasing of this theorem would be the following statement: If G is a subgraph of a 3-tree where every tetrahedron has an adjacent tetrahedra on at most 2 faces, then $c(G) \leq 3$. In other words, we want to forbid graphs like Figure 4.0.2 and allow graphs like Figure 4.0.1.

Proof. Have the cops start on the three vertices of a triangle that is adjacent to two tetrahedra. Call this triangle R_1. Now, one of these tetrahedra must be closer to the robber than the other one. Call the tetrahedron that is closer to the robber T_1. We already know that T_1 is adjacent to R_1, so T_1 must have at most one more face adjacent to other tetrahedra. If it does not, then the robber is caught. Let us call this other face R_2. The goal is to have the three cops occupy the three vertices of R_2. Two cops are already in place because T_1 has four vertices and the cops already occupy three of those vertices. Have the third cop that is in R_1 but not in R_2 move to empty vertex in R_2. Keep on following these steps until the cops are on a triangle that is adjacent to the tetrahedron that contains the robber and is not adjacent to any other tetrahedra. Then have the cops converge on the robber, and the robber is caught. □

5
Open Problems

There are three open problems that I would like to invite the reader to work on. The first two problems are to find proofs for the converses of two of theorems that I have found so far. The first theorem is a rephrasing of Theorem 4.0.4, which was proved in Chapter 4:

Theorem 5.0.5. *If G is a subgraph of a 3-tree where every tetrahedron has adjacent tetrahedra on at most 2 faces, then $c(G) \leq 3$.*

The first open problem is to find a proof of the converse of Theorem 5.0.5:

Conjecture 5.0.6. *If $c(G) \leq 3$, then G is a subgraph of a 3-tree where every tetrahedron has adjacent tetrahedra on at most 2 faces.*

From Chapter 3, it follows that the next theorem is true:

Theorem 5.0.7. *If $c(G) \leq 3$, then G does not have $FM_1, FM_2, ..., FM_7$ as a minor.*

Again, we would like to show that the converse of Theorem 5.0.7 is true. Finding a proof of the following conjecture is the second open problem:

Conjecture 5.0.8. *If G does not have $FM_1, FM_2, ..., FM_7$ as a minor, then $c(G) \leq 3$.*

To go about proving the second problem, I have attempted to prove instead the following statement:

Conjecture 5.0.9. *If G does not have $FM_1, FM_2, ..., FM_7$ as a minor, then G is a subgraph of a 3-tree where every tetrahedron has adjacent tetrahedra on at most 2 faces*

Then, by Theorem 5.0.5, it follows that if G does not have $FM_1, FM_2, ..., FM_7$ as a minor, then $c(G) \leq 3$.

Since the contrapositive of Conjecture 5.0.9 is easier to work with, I have attempted to show instead that if every 3-tree that G is a subgraph of has at least a tetrahedron that has tetrahedra on more than 2 faces, then G has $FM_1, FM_2, ..., FM_7$ as a minor.

To go about this problem, I have attempted to find all the subgraphs of a 3-tree where a tetrahedron has tetrahedra on three different faces as shown in Figure 4.0.1, and tried to show that these were either the subgraphs of a 3-tree where every tetrahedron has an adjacent tetrahedron on at most 2 faces as shown in Figure 4.0.2, or that they had one of $FM_1, FM_2, ..., FM_7$ as a minor. I have found every possible subgraph of Figure 4.0.1 by deleting every possible combination of edges and have concluded that this last statement is true. Unfortunately, the problem with this line of reasoning is that I have not considered the subgraphs of much larger graphs, larger than the graph shown in Figure 4.0.1, and am not likely able to do so as these possibilities are infinite. Therefore, the second open problem is to find a way to solve this problem.

The third open problem is to characterize graphs with cop number 4 or more, or even for cop number n. The difficulty with this problem is the sheer number of forbidden minors to find for cop number 4, as over 75 forbidden minors have been found for treewidth 4 [6]. On the other hand, Theorem 4.0.4 should generalize for larger cop numbers.

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