Abstract

This project is concerned with happy and sad numbers. To decide whether a number is happy or sad, you break it up into its individual digits, square each one, and add them all together. Then you take this result, and repeat the process. If you repeat the process enough times, you will eventually reach either 1, in which case your initial number is happy, or a set of 8 numbers that cycle endlessly, in which case it is sad. This fact is only true when we are working with numbers from the base-10 system that we use everyday. In other bases, such as the binary system that is used in computing, the function displays other cyclical behavior, such as cycles of varying lengths and fixed points other than 1. For some set bases and initial points, we can predict the eventual behavior of the function over time.
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Bibliography
Dedication

To my parents, for their constant encouragement, unconditional love, and financial support, without which this project would not have been possible.
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1

Introduction

1.1 Introduction

Dynamical systems is a branch of mathematics concerned with the behavior of functions over time. A dynamical system is a model that describes the temporal evolution of a system. It consists of a set of possible states that describe the values that the system may take, a time component, and a rule that determines the present state in terms of past states. Dynamical systems are characterized by sensitive dependence on initial conditions: because the rule is applied many times, iterations of initial points that were close together may end up far apart. Dynamical systems may be continuous or discrete. In continuous dynamical systems, the rule is established in terms of differential equations and the system is defined at every point on the interval. In discrete dynamical systems, the type we will address in this paper, the time component involves integers and the system evolves one step at a time.

Dynamical systems may also be either stochastic, involving some sort of probability, or deterministic, meaning that it must be possible to determine the present state uniquely.
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in terms of past states, without any element of randomness or chance. An example of a stochastic dynamical system would be the money you would make if I told you that I would give you five cents every time you flip 'tails' on a nickel, but take away five cents if you flip 'heads'. You would start out at zero, but each time you flip the coin you have an equal chance of losing or gaining the five cents. It is impossible to predict exactly how much money you would have after 5 flips—the possibilities could be described with a range between -$.25 and +$.25. In this paper, we will concern ourselves only with deterministic dynamical systems. In a deterministic dynamical system, the way the system changes is defined by a map, applied the same way every time.

The map is a function that assigns any element of the domain to exactly one element in the codomain. One example of such a rule would be \( f(x) = 2x \). This rule is the function that maps each number to a number twice as large. The various values for time are called iterations, and are described in terms of integers. In a dynamical system, we can use the rule to find the value of the output of the function over time. In the above function, if we arbitrarily pick the initial point \( x = 1 \), the first iteration is found by applying the rule once: \( f(1) = 2(1) = 2 \). The second iteration for initial point 1 is found by applying the rule again, to the output from the first iteration: \( f(2) = 4 \). Successive values for a system can be found by using function composition. For example, we can also write the second iteration for initial point 1 under \( f \) as \( f^2(1) = 4 \), with the 2 denoting that we are applying or 'iterating' the function twice. In general, we may find the \( m \)-th iteration of \( f \) by applying \( f \) \( m \) times, which we will denote \( f^m(x) \).

In this paper, we study the behavior of a particular discrete, deterministic dynamical system. The rule that defines this dynamical system is the function which sums the squares of the digits of an integer, and the behavior of an initial point under this function over time defines whether or not it is 'happy'. In the rest of this chapter, we will provide additional background information and key terms relating to dynamical systems. In Chapter 2, we
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define the rule that determines our particular dynamical system as well as what it is to be a 'happy number', and show the behavior of the function in our normal base 10 number system. In Chapter 3, we look at how the behavior of the function changes in bases other than 10, first examining base 2, then generalizing to an arbitrary base $b$, and providing a table with the behavior in bases 2-15. Chapter 4 tackles at the behavior of the function with a particular initial point $p$, and try to predict its behavior in bases $p^r$, starting with the simplest case where $p = 2$. Finally, Chapter 5 addresses possible topics for further research.

1.2 Background

When we study dynamical systems, we are interested in both the value of the function at the initial point and in the subsequent iterations. We can record the successive values of $f$ by finding the orbit. Let $x$ be a point in the domain of the map $f$.

**Definition 1.2.1.** The **orbit** of $x$ under $f$ is the set of points $\{x, f(x), f^2(x), f^3(x), \ldots\}$.

△

In the example we used above, $f(x) = 2x$, the orbit of 3 would be $\{3, 6, 12, 24, \ldots\}$. In this particular case, with initial point 3, we can see that each successive value is greater than the one before, so the value of the function is always changing. This is not always the case. Some dynamical systems have **fixed points** where, once they reach these, every subsequent value is identical.

**Definition 1.2.2.** A point $x_0$ is a **fixed point** of $f$ if $f(x_0) = x_0$.

△

The function $g(x) = x + 2$ has no fixed points, but if we use our function $f(x) = 2x$ from above, it is easy to see that 0 is a fixed point, because if we plug in $x = 0$, we get $f(0) = 2(0) = 0$. 


Some functions also have discrete sets of points where, once a number falls within the set, all subsequent points will also. For example, if we let \( h(x) = -x + 1 \), then we can see that \( h(0) = -0 + 1 = 1 \), and \( h(1) = -1 + 1 = 0 \). If some iteration of an initial value gives us 0 or 1, all subsequent iterations will alternate between 0 and 1.

**Definition 1.2.3.** Let \( f \) be a map, with \( x_0 \) a point in the codomain. If \( f^t(x_0) = x_0 \), and \( t \) is the smallest integer for which this is true, then \( x_0 \) is a **period-\( t \)** point. The orbit that begins with initial point \( x_0 \) and consists of \( t \) points is called the **period-\( t \)** orbit. Because of its cyclic behavior, we may also call this orbit a **\( t \)**-cycle. △

Our examples of 0 and 1 for \( h \) therefore are each period-2 points, because \( h^2(0) = 0 \) and \( h^2(1) = 1 \), and they form a period-2 orbit, also called a 2-cycle.

In many cases, a point \( x_0 \) may start out outside of a given \( t \)-cycle, but upon iteration, end up inside of it. The points that \( x_0 \) hits before one the period points are called **pre-periodic**. For example, if \( g(x) = x^2 - 1 \), then \(-1\) and \(0\) form a 2-cycle. But \( g(1) = 0 \), which is a point from the 2-cycle, so \(1\) would be pre-periodic.

Now that we have defined all of the important terms relating to dynamical systems that we are going to use, we can proceed to define the rule for the particular dynamical system that we are concerned with in this project.
2

Happy Numbers in Base 10

2.1 Introduction

In this chapter we will establish the rule for our particular dynamical system, which will help us to define what it is to be a happy or unhappy number. We will also address the interesting cyclic properties of happy and unhappy numbers.

2.2 Happy Numbers

Let \( n \) be a natural number with \( k \) digits, with each digit represented as \( a_i \), i.e. \( n = a_{k-1}a_{k-2}...a_1a_0 \). Then \( n \) can be rewritten as \( n = a_{k-1} \cdot 10^{k-1} + a_{k-2} \cdot 10^{k-2} + ... + a_1 \cdot 10 + a_0 \).

Let \( f \) be the function that squares each of the digits and adds them all together:

\[
f : \mathbb{N} \to \mathbb{N} \\
f(n) = \sum_{i=0}^{k-1} a_i^2.
\]

For example, let \( n = 3212 \). Then \( f(3212) = 3^2 + 2^2 + 1^2 + 2^2 = 18 \).

When this function is iterated, patterns of end behavior start to emerge:
2. **HAPPY NUMBERS IN BASE 10**

\[ f^1(3212) = 18 \]
\[ f^2(3212) = f^1(18) = 65, \]
\[ f^3(3212) = f^2(18) = f^1(65) = 61, \]
\[ f^4(3212) = f^3(18) = f^2(65) = f^1(61) = 37, \]
and so on.

There are two important sets that emerge from this function, \( H \) and \( S \). The first set, \( H \), consists of just the element 1.

**Lemma 2.2.1.** The set \( H = \{ 1 \} \) is a fixed point in base-10.

**Proof.** Starting at initial point 1, we take \( f(1) = 1^2 = 1 \). So \( f(1) = 1 \), and 1 is a fixed point for \( f \). \( \square \)

The second set, \( S \), consists of 8 elements: 4, 16, 37, 58, 89, 145, 42, and 20.

We will show that \( S \) is actually an 8-cycle, because once an iteration of an initial point falls within the set, successive iterations cycle through the ordered set of 8 elements.

**Lemma 2.2.2.** For all \( n \in S, m \in \mathbb{N} \), \( f^{(m)}(n) \in S \).

**Proof.** To demonstrate this, we will show that \( f(n) \) for any of these elements is also within \( S \). \( f(4) = 16, f(16) = 37, f(37) = 58, f(58) = 89, f(89) = 145, f(145) = 42, f(20) = 4 \). Therefore, any iteration of an element of \( S \) will also be in \( S \), and \( S \) is an 8-cycle. \( \square \)

We can generate any integer we wish under \( f \):

**Theorem 2.2.3.** For any \( y \in \mathbb{N} \), there are infinitely many integers \( n \) such that \( f(n) = y \).

**Proof.** For any \( y \), let \( v \) be a string of ones of length \( y \), so \( v = 1 \cdots 1 \), with \( k \) for \( v \) equal to \( y \). Then \( f(v) = 1^2 + 1^2 + \cdots + 1^2 \) for a total of \( k = y \) 1s, so \( f(v) = y \). This gives us at least one \( n \) for which \( f(n) = y \). To find additional numbers for which \( f(n) = y \), we may take \( v \) and add a digit of zero in any position except the first to give us a new \( v_0 \) with
$k = y + 1$. This addition adds an extra digit to $n$, but does not change the value of $f(n)$ because $f(0) = 0$. This procedure can be repeated, adding an infinite number of zeros without changing the value of $f(n)$. Each of these numbers satisfy $f(n) = y$, therefore there are infinitely many integers $n$ such that $f(n) = y$.

For example, if we let $y = 7$, then we can set $v = 1111111$. It is easy to see that $f(v) = 7$. We can also create $v_1 = 11111011$ and $v_2 = 1001110000111000000$ that also satisfy the equation.

Because of this property, any $x_m$ in a cycle has infinitely many possible pre-images, or previous steps, that are pre-periodic.

### 2.3 End Behavior for $f^{(m)}(n)$

It turns out that regardless of the initial $n$ to which you apply $f$, if you iterate the function enough times, $f^{(m)}(n)$ will eventually end up in either $H$ or $S$. This means that every natural number eventually ends up in one of our sets.

**Theorem 2.3.1.** For all $n \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ such that $f^{(m)}(n) \in S$ or $f^{(m)}(n) \in H$.

**Proof.** For any $n$, each digit has a maximum value of 9. The maximum value for $f(n)$ is achieved when every digit in $n$ is 9, so if $n$ has $k$ digits it occurs at $f(n) = 9^2(k)$. All other values of $f(n)$ are less than this, so $f(n) \leq 9^2k = 81k$. For all $n$ with $k$ digits, the first digit of $n$ must be non-zero, so $10^{k-1} \leq n < 10^k$. The expression $10^{k-1}$ increases exponentially while $81k$ increases only geometrically, and for all $k \geq 4$, $81k \leq 10^{k-1}$. We already know that $10^{k-1} \leq n$ and $f(n) \leq 81k$, so $f(n) \leq 81k \leq 10^{k-1} \leq n$, and $f(n) \leq n$ for $k \geq 4$, which means that the function is decreasing. So all the integers in base 10 for
2. HAPPY NUMBERS IN BASE 10

which $k < 4$ represent the only values for which $f(n)$ might be greater than $n$. The largest value for this occurs when $k = 3$, at $n = 999$, so $f(n) = 243$.

When $k \geq 4$, the function is decreasing, so for any $n > 243$ there exists some $m \in \mathbb{N}$ such that $f^{(m)}(n)$ is between 1 and 243. The greatest value of $f(n)$ on this interval is at 243. Therefore, to determine the eventual end behavior of all $n$, we must check only 1-243. See Appendix in 6.1 for the computer code that was used performed this check. Every $n \in [1, 243] \cap \mathbb{Z}$ converges to either $S$ or $H$, and all other $f(n)$ are decreasing and will land in $[1, 243] \cap \mathbb{Z}$, so all $n \in \mathbb{N}$ must converge to either $S$ or $H$. □

We now define what it is to be a happy number or a sad number.

**Definition 2.3.2.** Numbers that converge to $H$ are called happy numbers. △

**Definition 2.3.3.** Numbers that converge to $S$ are called sad numbers. △

So from the theorem above, all of the natural numbers converge to either $S$ or $H$, so they may all be classified as either happy or sad. We have already shown that 1 is a fixed point for $f$, but a corollary of Theorem 2.3.1 helps show that this is actually the only fixed point for $f$ in base 10.

**Corollary 2.3.4.** For all $n \in \mathbb{N}$, $f(n) = n$ if and only if $n = 1$.

**Proof.** We have shown that for all $n > 243$, $f(n)$ is a decreasing function, and therefore it suffices to check $[1, 243] \cap \mathbb{Z}$. If $f(n) = n$, then $f^2(n) = n$ and all further iterations will also be $n$. But we have shown that for all $n \in [1, 243]$, there exists some $m \in \mathbb{N}$ such that $f^{(m)}(n)$ is a member of $S$ or $H$. Therefore all $n$ go to $S$ or $H$, so the only fixed point is 1, and $f(n) = n$ if and only if $n = 1$. □
2. HAPPY NUMBERS IN BASE 10

In other words, no other number is the sum of the squares of its digits. As we will see later, this is not true in some other bases besides base-10. Another related corollary proves another interesting property:

**Corollary 2.3.5.** 1 is the unique base 10 number that is the sum of the squares of its digits.

**Proof.** The function $f$ sums the squares of the digits of a number $n$, therefore this statement is equivalent to saying that 1 is the only fixed point for $f$ in base 10. We have shown that for all $n > 243$, $f(n) < n$, and for a number to be a fixed point $f(n) = n$, so we can rule all of these out. We have also shown that all $n \in [1, 243] \cap \mathbb{N}$ go to either S or H. So 1 is the only fixed point for $f$ in base 10, and therefore 1 is the unique base 10 number that is the sum of the squares of its digits.

In addition to 1, there are several other constructions for initial points that are always happy:

**Theorem 2.3.6.** For all $r \in \mathbb{N}$, all numbers of the form $10^r$, $13(10^r)$, and $10^r + 3$ are happy.

**Proof.** If a number is of the form $10^r$, the initial digit is 1, followed by $r$ zeros. The square of zero is zero, therefore the sum of $r$ zeros is still zero, so $f(10^r) = 1^2 = 1$. If a number is of the form $13(10^r)$, the first two digits are 1 and 3, followed by $n$ zeros. As above, the sum of the squares of the $n$ zeros is zero, so to find $f(13(10^r))$ we add only $1^2 + 3^2 = 1 + 9 = 10$. 10 is of the form $10^r$, which we have already shown is happy. If a number is of the form $10^r + 3$, it looks like $10^r$ as we described above, except with the last digit being 3 instead of 0. The first digit is 1, followed by $r - 1$ zeros, and final digit 3. Again, we can ignore the sums of the squares of the $r - 1$ zeros, leaving us with $f(10^r + 3) = 1^2 + 3^2 = 10$, and
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$f(10) = 1^2 + 0^2 = 1$, so 10r + 3 is happy. Therefore all numbers of the form 10r, 13(10r),
and 10r + 3 are happy.

Once you show that a particular number is happy by iterating it under $f$ until it hits
1, all of the intermediate iterations are also happy.

**Lemma 2.3.7.** For all $w \in \mathbb{N}$, if $n$ is happy, $f^{(w)}(n)$ is also happy.

**Proof.** If $n$ is a happy number, then we have shown above that there exists some $m \in \mathbb{N}$
such that $f^{(m)}(n) = 1$. Let $n_0 = f^{(w)}(n)$. If $w > m$, we know that after $m$ iterations $f(n)$
has reached 1, so subsequent iterations will remain at 1. If $w < m$, then $f^{(m-w)}(n_0) =
f^{(m-w)}(f^{(w)}(n)) = f^{(m)}(n) = 1$. Therefore, if $n$ is happy, then any iteration of $f(n)$ is also
happy.

For example, if we figure out that 193 is happy, we know that $f^1(193) = 91$, $f^2(193) =
82$, $f^3(193) = 68$, and $f^4(193) = 100$ are all also happy. This also holds when $n$ is sad: for
$w \in \mathbb{N}$, all $f^{(w)}(n)$ are also sad, by the same argument.

Once you have found a few happy or sad numbers, there are several easy ways to generate
more. One way is by switching the digits around.

**Theorem 2.3.8.** If $n$ is a $k$ digit number with $n = a_0a_1a_2\cdots a_{k-1}$, then switching any
digits of $a_i$ will not affect whether $n \in H$ or $n \in S$.

**Proof.** Addition is commutative, therefore $f(n) = \sum_{i=0}^{k-1} a_i^2$ does not depend on the order
of the summation. Therefore switching the digits around will not affect $f(n)$, and therefore
will not affect whether $n$ is happy or sad.

From our example above, we know that 193 is happy. Therefore, 139, 319, 391, 913, and
931 are also happy. Another way to generate happy or sad numbers is by adding zeros to
them:
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Corollary 2.3.9. There are infinitely many happy and sad numbers.

Proof. Choose any happy or sad number \( n \) with length \( k \). We may add a zero between the 2nd and \( k - th \) digits of \( n \) without changing the value of \( f(n) \), giving us a new \( n_1 \) with \( k_1 = k + 1 \). The resulting number is also happy or sad, respectively. This procedure may be repeated infinitely, adding any number of zeros, therefore there are infinitely many happy and sad numbers.

Given that the natural numbers all eventually end up in \( H \) or \( S \) upon iteration under \( f \), it is not surprising that there is an infinite number of them, but it is important to note that this means that both happy and sad numbers can be infinitely large as well, i.e. that there is no boundary past which there are not more happy or sad numbers.
3

Happy Numbers in Other Bases

3.1 Introduction

The conclusions in the previous chapter about this function depended on the decimal representation of integers. The function can also be explored in other bases.

When we refer to a base $b$, the $b$ stands for the number of unique digits that are used to represent a number in that particular base. For base 10, we use 10 digits, 0-9. For base 6, however, we use only 6 digits, 0-5. For bases greater than 10, letters are often used to represent additional digits, with $A = 10$, $B = 11$, $C = 12$, and so on. Therefore base 15 uses digits 0 through 9 as well as $A, B, C, D$ and $E$.

When representing a number in any base $b$, we use a position system similar to the one we are used to with base 10. The right-most position represents $b^0$, which is 1 for any $b$, so it is the "ones" position. The position directly to the left of the ones position represents $b^1$, the one next to it is $b^2$, and so on.

The number 4902 in base 10 can be written as $4(10^3) + 9(10^2) + 0(10^1) + 2(10^0)$. The corresponding representation in base 8 is $1(8^4) + 1(8^3) + 4(8^2) + 6(8^1) + 0(8^0)$, or 11460$_8$. 
The value of the two expressions for 4902 add up to the same total, but it is represented in two different ways.

When we are working with large bases, we can have a string that represents a single digit or position. For example, if we want to write 4902 in base 500, we have $9(500^1) + 402(500^0)$. Writing this as $9402_{500}$ is confusing because we don’t know which digits correspond to which position. Instead, when we are working with bases larger than 36 and using more than one digit place, we will write it as $\{9, 402\}$ to make it clear which digits correspond to which places.

When working with other bases, the behavior of the function changes. In this chapter we will look at happy and unhappy numbers in different bases.

### 3.2 Base-2

In base-2, or binary representation, integers are represented by only 0s and 1s. If a $k$-digit integer has a 1 in a particular position $i$, counting from the left, it represents a value of $2^{k-i}$. For example, if we have $n = 1001_2$, with 1s in position 1 and 4 and $k = 4$, we can calculate $2^3 + 2^0 = 9$.

**Theorem 3.2.1.** All numbers in base 2 are happy.

**Proof.** Each digit of $n$ in base 2 has a maximum value of 1, so for any $n$ in base 2, $f(n) \leq k$, where $k$ is the number of digits of $n$. For each additional digit added, the value of $k$ increases by 1, while the maximum value of $n$ increases by powers of 2, so for all $k \geq 3$, we know that $k < n$, and $f(n) \leq k < n$. Therefore the function is decreasing for all $k \geq 3$, so we must check only $k = 1$ and 2 to find the behavior of any $n$ when iterated.

When $k = 1$, the only possibility for $n$ is $1_2$, and $f(1_2) = 1_2$ is a fixed point. When $k = 2$, the possibilities are $10_2$ and $11_2$:

$$f(10_2) = 1^2 + 0^2 = 1_2$$

$$f(11_2) = 1^2 + 1^2 = 10_2$$

$$f(10) = 10$$

$$f(11) = 11$$

Therefore, all numbers in base 2 are happy.
3. HAPPY NUMBERS IN OTHER BASES

\[ f(112) = 1^2 + 1^2 = 102 = 1^2 + 0^2 = 12 \]

Therefore for all \( k \leq 3 \), \( f(n) \) eventually reaches the fixed point 1. Therefore for all \( n \in \mathbb{N} \) in base 2, there exists some \( m \in \mathbb{N} \) such that \( f^m(n) = 1 \), and all numbers in base 2 are happy.

\[ \square \]

Definition 3.2.2. A happy base is a base in which all numbers, when iterated, eventually reach the fixed point 1.

\[ \triangle \]

We have already shown that every integer in base 2 is happy, therefore base 2 is a happy base.

3.3 Arbitrary Bases

We have seen how this function behaves in base-10 and base-2, but we can use many of the same techniques without specifying a particular base. In this section, we will look at generalizations that can be made for any arbitrary base \( b \).

We have seen that 1 is happy for bases 10 and 2, but it turns out this is true for every base:

Theorem 3.3.1. 1 is happy in every base.

Proof. Every base includes the digit 1, and 1 is the multiplicative identity in all bases. Therefore, regardless of the base, \( 1^2 = 1 \times 1 = 1 \), so 1 is happy.

\[ \square \]

We can also find infinitely many happy numbers in any base:

Theorem 3.3.2. There are an infinite number of happy numbers in every base \( b \).

Proof. We have already shown that 1 is happy in all bases. An arbitrary number of zeros can be added to the initial 1. These zeros do not change the value of \( f(n) \), so the integers
formed by adding these zeros is happy as well. This process can be repeated infinitely, for
infinite happy numbers in any base $b$.

Above we showed the critical regions of $\mathbb{N}$ for base 2 and base 10. Above the upper
bound of these regions, $f(n) < n$, so $f$ is decreasing and therefore all $n$ above the critical
region, upon iteration under $f$, will eventually end up within critical region. It is possible
to find the portion of $n \in \mathbb{N}$ for which $f(n)$ is not necessarily decreasing in a manner
similar to the procedure above.

The greatest digit in base $b$ is $b - 1$. Therefore, for any $n$, there is a maximum value of
$f(n)$: $f(n) \leq (b - 1)^2 k$, where $k$ is the number of digits. It is also true that $b^{k-1} \leq n < b^k$. 
So when $(b - 1)^2 k \leq b^{k-1}$, we have $f(n) \leq (b - 1)^2 k \leq b^{k-1} \leq n$, so $f(n) \leq n$. Therefore
we are concerned with the region $[1, (b - 1)^2 k]$.

When $b$ is known, it is easy to determine the maximum value of $k$ for such that $(b - 1)^2 k \leq
b^{k-1}$ and therefore $f(n)$ must be less than or equal to $n$. We will call this upper bound
for the critical interval $z$. For example, let $b = 8$. Plugging into our expression above, we
then have that $f(n) \leq n$ when $(b - 1)^2 k \leq b^{k-1}$, so when $7^2 k \leq 8^{k-1}$. We can see that this
expression is false for $k = 1$ ($49$ is not less than or equal to $1$), $k = 2$ ($98$ is not less than
or equal to $8$), and $k = 3$ ($147$ is not less than or equal to $64$). But it is true for $k = 4$ and
greater. Subtracting 1 from this $k$, you find the maximum $k$ for which the function may
be increasing. Let this $k = k_0$. In the case of base 8, this is 3. Then $z = (b - 1)^2 k_0$, and
by checking all $n \in [1, z]$, you can determine the end behavior for the entire function in
arbitrary base $b$. For base 8, we would check $[1, (7^2)3]$, or $1, 147$.

We have found the cyclical behavior for bases 2 and 10, and it turns out that any base
must have some sort of cyclic behavior.

**Theorem 3.3.3.** In base $b$, all elements, upon iteration, will eventually display cyclic
behavior, with maximum cycle length $z = (b - 1)^2 k_0$. 
3. HAPPY NUMBERS IN OTHER BASES

Proof. We have already shown above that for any base \( b \), we can find the critical region, \([1, z]\), which shows the end behavior for every possible integer in the base, because above \( z \) the function is decreasing. Therefore every element will eventually end up in the critical region. Let \( u \) be the number of iterations that it takes to get from an initial point \( x_0 \) to a point \( x_u \in [1, z] \). Because above \( z \) the function is strictly decreasing, subsequent iterations will never give us a point that is greater than \( z \). Because there are only \( z \) points in the critical interval, if we iterate \( x_u \) another \( z \) times, at least one element has to be repeated. Therefore, for any arbitrary base \( b \), we will find cyclic behavior, with maximum cycle length \( z = (b - 1)^2k_0 \).

\[
\]

3.4 Cycles in Bases 1 through 15

Using a computer program (see Appendix), it is possible to find all of the cycles for a particular base by checking the end behavior for all of the initial points within the critical region. All of the bases have happy numbers, and most have other cyclical behavior as well. We have found the end behavior for bases 2 and 10 above, here is the behavior of some other bases.
### 3. HAPPY NUMBERS IN OTHER BASES

<table>
<thead>
<tr>
<th>Base</th>
<th>Fixed Points</th>
<th>Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$1_3, 12_3, 22_3$</td>
<td>$(2_3, 11_3)$</td>
</tr>
<tr>
<td>4</td>
<td>$1_4$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$1_5, 23_5, 33_5$</td>
<td>$(45_5, 31_5, 20_5)$</td>
</tr>
<tr>
<td>6</td>
<td>$1_6$</td>
<td>$(32_6, 21_6, 5_6, 41_6, 25_6, 45_6, 105_6, 42_6)$</td>
</tr>
<tr>
<td>7</td>
<td>$1_7, 13_7, 34_7, 44_7, 63_7$</td>
<td>$(2_7, 4_7, 22_7, 11_7), (23_7, 16_7, 52_7, 41_7)$</td>
</tr>
<tr>
<td>8</td>
<td>$1_8, 24_8, 64_8$</td>
<td>$(4_8, 20_8), (32_8, 15_8), (5_8, 31_8, 12_8)$</td>
</tr>
<tr>
<td>9</td>
<td>$1_9, 45_9, 55_9$</td>
<td>$(75_9, 82_9), (58_9, 108_9, 72_9)$</td>
</tr>
<tr>
<td>10</td>
<td>$1_{10}$</td>
<td>$(410_{10}, 1610_{10}, 3710_{10}, 5810_{10}, 8910_{10}, 14510_{10}, 4210_{10}, 2010_{10})$</td>
</tr>
<tr>
<td>11</td>
<td>$1_{11}, 56_11, 66_11$</td>
<td>$(511_{11}, 2311_{11}, 1211_{11}), (6811_{11}, 9111_{11}, 7511_{11})$</td>
</tr>
<tr>
<td>12</td>
<td>$1_{12}, 25_{12}, A5_{12}$</td>
<td>$(512_{12}, 2112_{12}), (6812_{12}, 8412_{12}), (1812_{12}, 5512_{12}, 4212_{12}), (2212_{12}, 812_{12}, 5412_{12}, 3512_{12}, 2412_{12}, 8812_{12}, A812_{12}, 11812_{12}, 5612_{12}, 5112_{12})$</td>
</tr>
<tr>
<td>13</td>
<td>$1_{13}, 14_{13}, 36_{13}, 67_{13}, 77_{13}, A6_{13}, C4_{13}$</td>
<td>$(79_{13}, A0_{13}), (B2_{13}, 98_{13}), (53_{13}, 28_{13})$</td>
</tr>
<tr>
<td>14</td>
<td>$1_{14}$</td>
<td>$(61_{14}, 29_{14}), (8A_{14}, BE_{14}, 11B_{14}, 8B_{14}, D3_{14}, CA_{14}, 136_{14}, 34_{14}, AB_{14})$</td>
</tr>
<tr>
<td>15</td>
<td>$1_{15}, 78_{15}, 88_{15}$</td>
<td>$(9A_{15}, C1_{15}), (2_{15}, 41_{15}, 11_{15}), (12E_{15}, D6_{15}, DA_{15}), (8_{15}, 44_{15}, 22_{15}), (85_{15}, 5E_{15}, BE_{15}, 162_{15}, 2B_{15}), (15_{15}, 1B_{15}, 82_{15}, 48_{15}, 55_{15}, 35_{15}, 24_{15}), (57_{15}, 4E_{15}, 82_{15}, D5_{15}, CE_{15}, 17A_{15}, A0_{15}, 6A_{15}, 91_{15})$</td>
</tr>
</tbody>
</table>
3. HAPPY NUMBERS IN OTHER BASES

We can see that base 4, like base 2, is another happy base, because every number within the critical interval, upon iteration under $f$, ends up at 1.

Bases 3, 5, 7, 8, 9, 11, 12, 13, and 15 all have additional fixed points besides 1.

Bases 3, 8, 12, 13, 14, and 15 all have at least one 2-cycle, while bases 5, 8, 9, 11, 12, and 15 all have 3-cycles.

Base 7 is the only base with a 4-cycle, and base 15 is the only base with a 5-cycle or a 7-cycle.

Like base 10, base 6 also has an 8 cycle. Bases 14 and 15 both have 9-cycles, and base 12 has a 10-cycle, the longest cycle for any base up to 15.

The larger bases tend to have more different types of cyclic behavior. Base 2 has only one type (every number is happy, and goes to 1), while base 15 has three fixed points, one 2-cycle, three 3-cycles, and one cycle each of length 5, 7, and 9, for a total of ten different cycles. Base 12 is tied for the most different cycles, with seven fixed points and three 2-cycles.

Overall, it appears that the bases that are prime numbers tend to have shorter cycles than the bases that are composite numbers. The bases with cycles of length 5 through 10 are all composites, while the longest cycles for a prime base are the two 4-cycles in base 7.
4

Paths of Initial Point $p$ in Bases $p^r$

4.1 Introduction

In this chapter we will define not only the bases in which we are working, but also the initial points that we are iterating. We are looking, in general, at paths of the initial point $p$ in base $p^r$, but we will begin by focusing on the most simple case: when $p = 2$.

4.2 Initial Point 2 in Bases $2^r$

We start with bases of the form $2^r$, where $r \in \mathbb{N}$, and initial point 2. We can rewrite any $r$ as $2^c + d$ for some $c \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$ such that $2^c \leq 2^c + d < 2^{c+1}$. For example, if $r = 18$, then $c = 4$ and $d = 2$, and if $r = 32$, $c = 5$, and $d = 0$. It turns that for all such numbers, where $d = 0$, the initial point 2 is happy.

**Theorem 4.2.1.** For all $c \in \mathbb{N}$, in all bases of the form $b = 2^{2^c}$, 2 is happy.

**Proof.** We start with initial point $x_0 = 2$, or $2^1$. Squaring is equivalent to doubling the exponent, so we get $x_1 = 2^2$, and then $x_2 = 2^4$ and so on, with each successive point
4. PATHS OF INITIAL POINT P IN BASES $P^R$

$x_m$ in the path of 2 having the form $2^{2m}$. Eventually, we reach $x_c$, where $x_c = 2^{2^c} = b$, therefore the $x_c$ in base $b$ is represented as having a 1 in the $2^{2^c}$ place, and zeros elsewhere ($x_c = \{1, 0\}$). This single 1 is squared in the next iteration to give us a 1 in the ones place, so initial point 2 is happy for all bases of the form $b = 2^{2^c}$.

For all $r$ where $d \neq 0$, we can make generalizations about the first $c + 1$ iterations of 2.

**Theorem 4.2.2.** For any $c, d \in \mathbb{N}$, the first $c + 1$ iterations of the initial point $x_0 = 2^1$ are of the form $2^{2^1}, 2^{2^2}, \ldots, 2^{2^c}, \{2^{2^c-d}, 0\}$.

**Proof.** We are iterating the initial point 2, so we start with initial point $x_0 = 2$, or $2^{2^0}$. Squaring this, we get $x_1 = 2^{2^1}$, and then $x_2 = 2^{2^2}$ and so on, with each successive point $x_m$ in the path of 2 having the form $2^{2m}$. The first $c$ iterations proceed like this, so we have $2^1, 2^2, 2^4, \ldots, 2^{2^c}$. After this $c$-th step, squaring $2^{2^c}$ gives us $2^{2^{c+1}}$. Because we defined $2^c + d < 2^{c+1}$, we know that $2^{2^{c+1}} > 2^{2^c+d}$. Therefore this value is greater than the base, so we must use the second position to represent the number. We want to find the greatest number of times that $2^{2^c+d}$ goes into $2^{2^{c+1}}$ so we have $2^{2^{c+1}} / 2^{2^c+d}$, or equivalently, $2^{2^{c+1} - 2^c - d} = 2^{2^c-d}$. This goes in the second position, giving us $\{2^{2^c-d}, 0\}$ for iteration $c + 1$.

For example, if we are looking at the base with exponent $r = 2^c + d$ such that $c = 4$ and $d = 1$, then the base is $2^{17}$ and the first $c + 1 = 5$ iterations are as follows: $2, 2^2, 2^4, 2^8, 2^{16}, \{2^{15}, 0\}$. We can also make a few generalizations about the format of each iteration. The first is that applying $f$ to any $x = 2^w$ does not affect the base $p = 2$, but only the exponent.

**Lemma 4.2.3.** All digits of $f^m(2)$ in base $2^r$ with $r \in \mathbb{N}$, will be of the form $2^v$ for some $v \in \mathbb{N}$. 

4. PATHS OF INITIAL POINT P IN BASES $P^R$

**Proof.** There are two cases. If $d = 0$, then the base is of the form $2^{2^v}$ and is happy by Theorem 4.2.1. Therefore the elements have the form $2^{2^0}, 2^{2^1}, \ldots, 2^{2^{v-1}}, \{2^0, 0\}$, with the last element repeating indefinitely. Therefore when $d = 0$, all of the digits of $f^m(2)$ are of the form $2^v$ for some $v \in \mathbb{N}$.

If $d > 0$, then the base is of the form $2^{2^v+d}$. Therefore, by Theorem 4.2.2, the first $c+1$ iterations of the initial point $2^1$ are of the form $2^{2^1}, 2^{2^2}, \ldots, 2^{2^c}, \{2^{2^c-d}, 0\}$.

Each subsequent iteration involves first doubling the exponent, then deciding if this exponent is less than $r$. Call this doubled exponent $2x$. If $2x < r$, then we get $2^{2x}$. If $2x > r$, then the value is greater than the base and we must divide the value by the base: $2^{2x}/2^r = 2^{2x-r}$. Subsequent iterations repeat the process, replacing the new iteration for $x$. Therefore all digits are of the form $2^{2x}$ or $2^{2x-r}$ for some previous exponent $x$, so for $d > 0$ all digits are of the form $2^v$. We have shown this to be true for both cases, so all digits of $f^m(2)$ in base $2^r$ with $r \in \mathbb{N}$, will be of the form $2^v$ for some $v \in \mathbb{N}$. 

The second is that iterations will never use more than one digit slot at a time. Recall from Section 3.1 that for bases other than 10, the rightmost digit represents $b^0 = (2^r)^0 = 1$ the second digit is $b^1 = (2^r)^1 = 2^r$, the third digit is $b^2 = (2^r)^2 = 2^{2r}$, and so on.

**Lemma 4.2.4.** For all iterations $f^m(2)$ with $m \in \mathbb{N}$ in base $2^r$, $f(n)$ will only ever use one digit slot at a time.

**Proof.** Let the base $b = 2^r$. We have shown in Lemma 4.2.3 that all digits of $f^m(2)$ are of the form $2^v$ for some $v \in \mathbb{N}$. This means that each iteration is of the form $x_i = \{2^{v_1}, 2^{v_2}, \ldots, 2^{v_i}\}$ for some $i \in \mathbb{N}$. Taking $f(x_i)$, we have $2^{2^{v_1}} + 2^{2^{v_2}} + \cdots + 2^{2^{v_i}}$. But summing a series of terms with the same base with different exponents is equivalent to taking the base with the sum of all of the exponents, so we have $x_{i+1} = 2^{2^{v_i} + 2^{v_2} + \cdots + 2^{v_i}}$. This term is either less than the base $2^r$, in which case it is only in the rightmost digit slot and satisfies the property, or it is greater than the base, in which case we have
to find out how many times the base goes into $x_{i+1}$ by division. But dividing by a number with the same base is the same as subtracting one base from another, so we get $2^{2(v_1)+2(v_2)+\cdots+2(v_i)-r}$. But this means that the base $2^r$ goes into $x_{i+1}$ a total of $2^{2(v_1)+2(v_2)+\cdots+2(v_i)-r}$ times, without remainder. If it turns out that $2^{2(v_1)+2(v_2)+\cdots+2(v_i)-r}$ is also greater than $b^2 = 2^{2r}$, or even $2^{3r}$ or more, then we subtract $2r$ or $3r$ instead from the exponent, which still leaves us with no remainder. Therefore, $f^m(2)$ will only ever use one digit slot at a time.

The third is that any iteration of $f^m(2)$ will only use either the first slot, representing $b^0 = 1$, or the second slot, representing $b^1 = 2^r$, and never any higher slots.

**Lemma 4.2.5.** For all iterations $f^m(2)$ with $m \in \mathbb{N}$ in base $2^r$, $f(n)$ will only ever use the first and second digits for representation.

**Proof.** The greatest digit used in a particular base is $b - 1$, so the greatest digit in base $2^r$ is $2^r - 1$. But we have shown in Lemma 4.2.3, that all digits of iterations of $f$ have the form $2^v$ for some $v \in \mathbb{N}$, so the greatest value of $f^m(2)$ for each slot is $2^{r-1}$. We have also shown in Lemma 4.2.4 that iterations of $f^m(2)$ will only use one digit slot at a time. Therefore, $f(n) \leq f(2^{r-1}) = 2^{(2r-2)}$. The third digit represents $b^2 = 2^{2r}$. But we have already shown that the greatest value of $f(n)$ is $2^{(2r-2)}$, and $2r - 2 < 2r$, so $f(n) < b^2$ and will therefore never use the third slot. Therefore any iteration of $f^m(2)$ can be represented using only the first and second digit slots.

These properties help us to begin to constrain the possible cycle lengths for base $2^r$.

**Theorem 4.2.6.** In any base of the form $2^r$, the maximum cycle length for initial point 2 is $r - 1$. 

4. PATHS OF INITIAL POINT P IN BASES \( P^R \)

**Proof.** Let \( b \) be a base of the form \( 2^r \). We know that the maximum value for either digit is \( 2^{r-1} \). This means that there are \( r - 1 \) possibilities for the rightmost digit and \( r - 1 \) possibilities for the second digit. But any digit in the right position represents the square of the previous iteration, and therefore must be even, so we rule out half of the possibilities for the rightmost digit. There are two cases: if \( r \) is odd, we have \( \frac{r-1}{2} \) possibilities for the right digit, while if \( r \) is even, we have \( \frac{r}{2} - 1 \) possibilities for even numbers other than \( r \).

Any digit in the left position represents the square of the previous iteration divided by the base, which is equivalent to subtracting the exponents. The exponent from squaring a previous iteration must necessarily be an even number. If the exponent for the base is an odd number, we have only odd possibilities, not including \( r \), for the leftmost position for a total of \( \frac{r-1}{2} \) possibilities. If the exponent for the base is even, we again have only even possibilities, for a total of \( \frac{r}{2} - 1 \). We know that a particular iteration must either have a left digit or a right digit, giving us a total of \( \frac{r-1}{2} + \frac{r-1}{2} = r - 1 \) possibilities for values of \( f^m(2) \) with \( m \in \mathbb{N} \) when \( r \) is odd, or \( \frac{r}{2} - 1 + \frac{r}{2} - 1 = r - 2 \) possibilities when \( r \) is even.

Because there are only a limited number of possible values (\( r - 1 \) for odd \( r \) and \( r - 2 \) for even \( r \)) and we know that the function eventually reaches cyclic behavior, the max length of a cycle for base \( 2^r \) with \( r \) odd is \( r - 1 \). The maximum cycle length for base \( 2^r \) with even \( r \) is \( r - 2 \), but this is less than \( r - 1 \), so the maximum cycle length still holds.

For any \( r \), there is some \( q \in \mathbb{N}, c \in \mathbb{N} \cup \{0\} \) such that \( r = q(2^c) \). For example, we can rewrite \( r = 304 \) as \( r = 19(2^4) \). For odd \( r \), \( q = r \) and \( c = 0 \). We can make generalizations about the cycles of 2 given \( q \). For example, we can look at when \( q = 3 \).

**Theorem 4.2.7.** When \( q = 3 \), and therefore the base is \( 2^{3(2^c)} \), the eventual cycle length for initial point 2 is 2.

**Proof.** Let 2 be an initial point iterated under \( f \) in base \( 2^{3(2^c)} \). The initial point 2, or \( 2^{20} \) goes to \( 2^{21} \), which goes to \( 2^{22} \), until such point when \( 2^{2^m} > 2^{3(2^c)} \), but \( 2^{2^{m-1}} < 2^{3(2^c)} \) or
equivalently, \(2^{m-1} < 3(2^c) < 2^m\). Because \(2^1 < 3 < 2^2\), it follows that \(2^1(2^c) < 3(2^c) < 2^2(2^c)\), or equivalently, \(2^{c+1} < 3(2^c) < 2^{c+2}\). So if we let \(m = c + 2\), then we have the desired \(2^{m-1} < 3(2^c) < 2^m\). Therefore on the \(c + 2\) step, the result of applying \(f\) is greater than the base, so we must divide, getting \(2^{2c+2} / 2^3(2^c) = 2^{2c+2-3(2^c)} = 2^{2^2(2^c)-3(2^c)} = 2^{2c}\). Because we have set \(m = c + 2\), we have already passed through step \(c\), which we know proceeds to step \(c + 1\). But then step \(c + 2\) goes back to step \(c\), so the function oscillates between steps \(c\) and \(c + 1\). So for initial point 2 in bases of the form \(2^{3(2^c)}\), we have a cycle of length 2.

This can be generalized for any \(q\):

**Theorem 4.2.8.** For bases of the form \(2^{q(2^c)}\), with \(q \in \mathbb{N}\), bases with the same \(q\) will also have the same cycle length for the eventually periodic behavior of the initial element 2.

**Proof.** We start with initial point \(2^{2^0}\), which goes to \(2^{2^1}\), with each successive point having the form \(2^{2^m}\), until such point when \(2^{2^m_0} > 2^q(2^c)\), or equivalently, \(2^{m_0} > (2^c)q\). Let this point \(p = \lceil \log_2 q \rceil\). Therefore \(q < 2^p\), so \((2^c)q < (2^c)(2^p)\). Therefore this \(m_0 = c + p\). So after \(c + p\) steps, the element will be of the form \(\{2^{2^{c+p}-2^c(q)}, 0\}\), or \(\{2^{2^c(2^p-q)}, 0\}\). For each subsequent step, we double the exponent, decide whether this resultant exponent is larger than the base, and if it is we subtract \((2^c)q\). Therefore step \(c + p + x\) where \(x \geq 1\) will be of the form \((2^c)[2^{p+x} - (2^x + \alpha)]\). The term \(\alpha\) is a summation of \(2^{x_i-1}\) for all \(x_i\) where the exponent of the state is greater than the exponent of the base. Regardless of which case we choose, each option for the exponent has the same leading factor \(2^c\) because the base has a factor of \(2^c\). Because every element contains this factor in the exponent, it does not change over time, and thus does not affect the cyclical behavior of the function: the eventual cyclic behavior of the elements will not be affected by the term \(2^c\) except that the behavior happens later for larger \(c\). Therefore, the factor of \(2^c\) can be removed from each element and the base for any \(c\) to get the same cyclical behavior, so bases with the
same $q$ will also have the same cycle length for the eventually periodic behavior of the initial element 2.

Because the cycle length for initial point 2 in bases $2^{c(2^c)}$ is the same for all $c$, to find the cyclic behavior of any $q$ it is sufficient to check when $c = 0$, base $2^d$.

For example, we can look at the cyclic behavior for initial point 2 in base 6684672. This looks very complicated, but we can factor it: $6684672 = (17)(3)(2^{17})$. According to Theorem 4.2.8, this means that the cyclic behavior for 6684672 should be identical to $(17)(3) = 51$, which is much easier to calculate.

This also helps us constrain the maximum cycle length for bases with even exponents:

**Theorem 4.2.9.** In any base of the form $2^r$ where $r$ is even, the maximum cycle length for initial point 2 is $\frac{r}{2} - 1$.

**Proof.** Let $b$ be a base of the form $2^r$ such that $r$ is even. This means that $r = 2k$ for some $k \in \mathbb{N}$. Because $2 = 2^1$, we know from Theorem 4.2.8 that initial point 2 has the same cycle length in base $2^{2k}$ as it does in $2^k$. We know from Theorem 4.2.6 that the maximum cycle length for 2 in this base is $k - 1$. We can rewrite $r = 2k$ to get $k = \frac{r}{2}$. Therefore the maximum cycle length in a base of the form $2^r$ where $r$ is even is $\frac{r}{2} - 1$.

Here is the cycle length for all simplified $q$ from 1 to 100.
4. **PATHS OF INITIAL POINT P IN BASES $P^R$**

<table>
<thead>
<tr>
<th>Cycle Length</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
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<tr>
<td>4</td>
<td>5, 15</td>
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<td>82</td>
<td>83</td>
</tr>
</tbody>
</table>
4. PATHS OF INITIAL POINT P IN BASES $P^R$

4.3 Generalization to Bases $p^r$

**Theorem 4.3.1.** For base $p^r$ and initial point $p$, Theorems 4.2.1, 4.2.2, 4.2.6, 4.2.8, and 4.2.9, as well as Lemmas 4.2.3, 4.2.4, and 4.2.5 for base $2^r$ are also true if you change the 2 that is the base to the exponent in base $2^r$ and the initial point to the appropriate $p$.

**Proof.** The proof of each of these statements involves no manipulation of $p$, but only of the exponents $r$. Therefore the statements hold with any arbitrary $p$. □
Future Research

For future research, there are several conjectures for which I found extensive evidence, but that I was not able to prove. The first predicts the eventual cycle length for initial point 2 for all bases where $d = 2^j$, i.e. of the form $2^{2c+2j}$.

**Conjecture 5.0.2.** For all bases of the form $2^{2c+2j}$ with $c, j \in \mathbb{N}$ and $c > j$, the eventual cycle length for initial point 2 is $2(c - j)$.

We know that $c > j$, so we may rewrite: $2^{2c+2j} = 2^{(2j)(2c-j+1)}$. When $c - j = 1$ the cycle length should be $2(c - j) = 2$. Plugging in, we have $2^{(2j)(2^1+1)} = 2^{(2j)(3)}$ and we have already shown in Theorem 4.2.7 that the cycle length for bases of this form is 2, so the conjecture holds for this case.

Plugging in large values for $c$ and $j$ predict the cycle lengths found by my computer program from the Appendix. For example, when $c = 13$ and $j = 6$, then $r = 8256$. By Theorem 4.2.8, we can factor out a $2^6$, because $8256 = (2^6)(129)$, and from Theorem 4.2.6 we can say that the cycle length will be less than 128, but the computer program gives a precise answer of 14, which is the same as $2(13) - 2(6) = 14$, as the formula predicts. Despite this evidence, so far I have not been able to prove that this conjecture is true.
5. FUTURE RESEARCH

I have proved that the limit for cycle length for initial point 2 in bases $2^r$ is $r - 1$, but it seems that the only $r$ for which the cycles are actually that long are primes with primitive root 2. All of the bases with cycles of length $\frac{r}{2} - 1$ were of the form $2^{2x}$ for some $x$ from the previous set. It would be interesting to investigate why these patterns occur.

Also, though I found cycles for bases 1-15, I was not able to predict the length or content of these cycles. I made comments about the size of $b$ versus the length of cycles, as well as length of cycles for prime versus composite numbers, but this was based only on my observations. The same is true for values of $q$ in bases of the form $2^q(2^n)$. I was able to calculate individual cycles, but not predict what they would look like beforehand. Also, I noticed that multiplying $q$ by 3 often led to cycles of the same length, but this was not true in every case. It would be interesting to find the missing connections here.
6.1 Finding Possible Cycles for a Given Base

To find all of the possible cycles for various bases, I used Mathematica. The following code was used, with $b$ replaced with the appropriate base, and $z$ by the upper bound for the critical region of the particular base found using the procedure described in Section 3.3.

\[
f[x_\_] := \text{Total[IntegerDigits[x, b]^2]}
\]
\[
g[x_\_] := \text{NestWhileList[f, x, UnsameQ, All]}
\]
\[
h[n\_] := \text{BaseForm}[n, b]
\]

\[
\text{TableForm[Table[Table[ h[n], \{n, g[p]\}], \{p, 1, z\}]]}
\]

This program gives the behavior for all numbers in the critical region, from which we can find the possible behaviors of the function in that particular base.

The function $f[x]$ sums up the squares of the digits in the proper base, then $g[x]$ iterates this function until there is a value that is repeated. The upper bound for the critical region
6. APPENDIX

$z$ is described in base 10, so the function $h[n]$ converts the values into the correct base. The last line creates a table with left hand column having initial points 1 through $h[z]$, and then the iterations of these numbers until they repeat themselves.

6.2 Finding Cycles for a Given Initial Point

To find the cycles of a point 2 in base $2^r$, I used the following program, with $r$ replaced by the appropriate integer exponent:

\[
g[x_] := \text{Piecewise}\{\{2 \times, 2 \times < r\}, \{2 \times - r, 2 \times > r\}\}
\]
\[
h[x_] := \text{NestWhileList}[g, x, \text{UnsameQ}, \text{All}]
\]

Because the base is not affected by squaring, we are working with only the exponent $r$, so each successive value $x_i$ actually represents $2^{x_i}$. The function $g[x]$ represents squaring the value, which is equivalent to doubling the exponent. If the resultant value $x$ for the exponent is less than the base exponent $r$ then we know $2^x < 2^r$, so it prints this value. Otherwise it prints $2x - r$, or the exponent found when the resultant value is divided by the base. In the second line, $h[x]$ iterates this until there is a value that is repeated. The last line sets the initial value: because we are looking at initial point $2 = 2^1$, we enter $x = 1$. 