Series With Negative Terms
Summary of Section 8.4 (Part B)

Series with some negative terms can behave very differently than series whose terms are all positive. For a positive series \( \sum a_n \), there are only two possibilities:

1. \( \sum a_n \) converges, or
2. \( \sum a_n = \infty \) (and hence \( \sum a_n \) diverges).

As the following example shows, a series with negative terms can diverge without adding up to infinity.

**Example 1** Consider the following series:

\[
\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots
\]

The partial sums are:

\[
\begin{align*}
  s_1 &= 1 \\
  s_2 &= 1 - 1 = 0 \\
  s_3 &= 1 - 1 + 1 = 1 \\
  s_4 &= 1 - 1 + 1 - 1 = 0 \\
  &\vdots
\end{align*}
\]

As you can see, the sequence of partial sums oscillates between 0 and 1, having no limit as \( n \to \infty \). Therefore, the series diverges.

Perhaps you are thinking that the sum should be 0, since the 1's and \(-1\)'s cancel in pairs:

\[
\begin{align*}
  1 - 1 + 1 - 1 + 1 - 1 + \cdots &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\
  &= 0 + 0 + 0 + \cdots \\
  &= 0
\end{align*}
\]

However, there is just as good an argument that the sum should be 1:

\[
\begin{align*}
  1 - 1 + 1 - 1 + 1 - 1 + \cdots &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\
  &= 1 + 0 + 0 + 0 + \cdots \\
  &= 1
\end{align*}
\]

The point is, there really is no way to make sense of this infinite sum. It is neither infinity nor a number. It just diverges.
Here is a picture of the series in the last example:

![Series Diagram]

As you add the terms of the series together, the partial sums repeatedly jump back and forth (oscillate) between 0 and 1. This is a new kind of divergence: \textit{divergence by oscillation}.

Not every series that “jumps back and forth” diverges. Consider the series:

\[
1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots
\]

This is a geometric series with \( r = -\frac{1}{2} \) and \( a = 1 \), so the sum is:

\[
\frac{a}{1 - r} = \frac{1}{1 - (-1/2)} = \frac{1}{3/2} = \frac{2}{3}
\]

Here is a picture of this series:

![Series Diagram]

As you can see, the partial sums repeatedly jump back and forth across 2/3. The jumps get smaller and smaller, so the series converges to 2/3.
Partial sums are much more important for series with negative terms than they are for positive series. For a positive series, it is reasonable to imagine all of the terms being added together simultaneously. For a series with negative terms, you really must add the terms one at a time, with the sum of the series being the limit of the partial sums.

**PARTIAL SUMS**

The **partial sums** of the series \( \sum_{n=1}^{\infty} a_n \) are the numbers:

\[
\begin{align*}
    s_1 &= a_1 \\
    s_2 &= a_1 + a_2 \\
    s_3 &= a_1 + a_2 + a_3 \\
    s_4 &= a_1 + a_2 + a_3 + a_4 \\
    &\vdots
\end{align*}
\]

The sum of the series is by definition the limit of the partial sums:

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n
\]

It is usually easiest to compute the partial sums of a series recursively, using the formulas:

\[
\begin{align*}
    s_1 &= a_1 \\
    s_2 &= s_1 + a_2 \\
    s_3 &= s_2 + a_3 \\
    &\vdots
\end{align*}
\]

**EXAMPLE 2**  Find the sum of the following series:

\[
1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots
\]

**SOLUTION**  The partial sums are:

\[
\begin{align*}
    s_1 &= 1 \\
    s_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
    s_3 &= \frac{3}{2} - \frac{1}{2} = 1 \\
    s_4 &= 1 + \frac{1}{3} = \frac{4}{3} \\
    s_5 &= \frac{1}{3} - \frac{1}{3} = 0 \\
    s_6 &= 0 + \frac{1}{4} = \frac{1}{4} \\
    s_7 &= \frac{1}{4} - 1 = -\frac{3}{4} \\
    &\vdots
\end{align*}
\]

This sequence of numbers approaches 1, so the sum of the series is 1.
The importance of partial sums cannot be overstated. For a series with negative terms, the additions must occur one at a time, in order. Indeed, mathematicians have found that you can often change the sum of a series by rearranging the order of the terms!

Another important principle is that a series whose terms don't go to zero must diverge:

**DIVERGENCE TEST**

If \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

The reasoning behind this test is not as obvious in general as it is for positive series. If the terms of a positive series don't go to zero, the sum is clearly infinite. For a general series, the sum might be infinite, or the series might diverge by oscillation. The important point is that the terms of a convergent series always go to zero.

**EXAMPLE 3** Determine whether the following series converges or diverges:

\[
\frac{1}{2} - \frac{2}{3} + \frac{2}{3} - \frac{3}{4} + \frac{3}{4} - \frac{4}{5} + \cdots
\]

**SOLUTION** This looks similar to a telescoping series, but there is a subtle difference.

For a telescoping series, each term of the series is a difference of two numbers:

\[
\left( \frac{1}{2} - \frac{2}{3} \right) + \left( \frac{2}{3} - \frac{3}{4} \right) + \left( \frac{3}{4} - \frac{4}{5} \right) + \cdots
\]

For example, the first term is \( \frac{1}{2} - \frac{2}{3} = -\frac{1}{6} \), the second term is \( \frac{2}{3} - \frac{3}{4} = -\frac{1}{12} \), and so on. This is necessary to make the partial sums converge:

\[
s_1 = \frac{1}{2} - \frac{2}{3} \quad s_2 = \frac{1}{2} - \frac{3}{4} \quad s_3 = \frac{1}{2} - \frac{4}{5} \quad \cdots
\]

This telescoping series converges to \( 1/2 - 1 \), which is \(-1/2\).

The series we were given is not telescoping, because it does not have the necessary parentheses. The first term of the given series is just \( \frac{1}{2} \), the second term is just \( -\frac{2}{3} \), and so on. Indeed, the terms of the given series don't even approach zero, so the series **diverges** by the Divergence Test.

Not convinced? Here are the partial sums for the given series:

\[
s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2} - \frac{2}{3} \quad s_3 = \frac{1}{2} \quad s_4 = \frac{1}{2} - \frac{3}{4} \quad s_5 = \frac{1}{2} \quad s_6 = \frac{1}{2} - \frac{4}{5} \quad \cdots
\]

As you can see, the odd-numbered partial sums are all \( \frac{1}{2} \), while the even-numbered partial sums are getting closer to \(-\frac{1}{2}\). As a result, the series oscillates between \( \frac{1}{2} \) and \(-\frac{1}{2}\).
The last example was similar to example 1, in that the placement of the parentheses can affect the sum of the series. Indeed, if we place the parentheses differently, we can get the series to converge to positive 1/2:

$$\frac{1}{2} + \left( \frac{2}{3} + \frac{2}{3} \right) + \left( -\frac{3}{4} + \frac{3}{4} \right) + \left( -\frac{4}{5} + \frac{4}{5} \right) + \cdots$$

$$= \frac{1}{2} + 0 + 0 + 0 + \cdots$$

$$= \frac{1}{2}$$

**Absolute Convergence**

Consider the following series:

$$1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

This is similar to the geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$, except that every third term has been negated. Do you think this series converges?

Of course it does. If the terms of a series go to zero quickly enough, it shouldn't matter whether they are being added or subtracted. The terms of the above series become very small, very quickly, which precludes any sort of divergence (including divergence by oscillation).

In general, a series with negative terms can be transformed into a positive series by changing all the minus signs to plus signs:

$$1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \cdots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

(This is the same as taking the absolute value of all the terms.) If the terms are small enough that the positive series converges, then the original series must converge as well.

**ABSOLUTE CONVERGENCE TEST**

A series $\sum a_n$ **converges absolutely** if the associated positive series $\sum |a_n|$ converges.

Any series that converges absolutely must converge.
EXAMPLE 4  Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges or diverges.

SOLUTION  The $(-1)^n$ causes the signs of the terms to alternate:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \cdots$$

This is because $(-1)^n$ is an alternating sequence of 1's and -1's:

$$(-1)^1 = -1 \quad (-1)^2 = 1 \quad (-1)^3 = -1 \quad (-1)^4 = 1 \quad \cdots$$

Therefore, taking the absolute value just eliminates the $(-1)^n$:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{|(-1)^n|}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges absolutely.  

Keep in mind that that the absolute convergence test is *inconclusive* when the positive series $\sum_{n=1}^{\infty} |a_n|$ diverges.  For example, the series:

$$1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots$$

converges (see example 2), but it clearly doesn't converge absolutely.

Sometimes it can be difficult to determine whether the associated positive series converges.  The following example requires the Comparison Test:

EXAMPLE 5  Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ converges or diverges.

SOLUTION  This series has negative terms because of the $\sin n$ in the numerator.  The associated positive series is:

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$$

It is not immediately clear whether this series converges.

We use the Comparison Test.  Observe that

$$|\sin n| \leq 1$$

for all $n$, and therefore:

$$\frac{|\sin n|}{n^3} \leq \frac{1}{n^3}$$
Since \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges, the smaller series \( \sum_{n=1}^{\infty} \frac{|\sin n|}{n^3} \) must converge as well, and therefore the series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^3} \) converges absolutely.

Note that \( \sum_{n=1}^{\infty} \frac{\sin n}{n^3} \) cannot be compared directly with \( \sum_{n=1}^{\infty} \frac{1}{n^3} \), since the first series has negative terms. The Comparison Test can only be applied to positive series.

The following example uses the root test.

**EXAMPLE 6** Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n^3 e^n} \) converges or diverges.

**SOLUTION** The absolute value of \((-2)^n\) is just \(2^n\):

\[
\sum_{n=1}^{\infty} \left| \frac{(-2)^n}{n^3 e^n} \right| = \sum_{n=1}^{\infty} \frac{2^n}{n^3 e^n}
\]

This series converges by the root test (with \( r = 2/e < 1 \)), and therefore the series \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n^3 e^n} \) converges absolutely.

Suppose that the last example had been slightly different, so that \( r \) had come out greater than one. Would this prove that the series diverges?

Yes, though this does not follow from the Absolute Convergence Test. It is instead related to an important property of the root test: **if a series has \( r > 1 \), then the individual terms of the series go to infinity.** The reason that a series with \( r > 1 \) diverges is that it fails the Divergence Test, and this is not affected by whether the terms are positive or negative.

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**THE ROOT TEST (FOR GENERAL SERIES)**

Let \( \sum a_n \) be any series, and let:

\[
r = \lim_{n \to \infty} \sqrt[n]{|a_n|}
\]

(a) If \( r < 1 \), then the series \( \sum a_n \) converges absolutely.

(b) If \( r > 1 \), then the series \( \sum a_n \) diverges.

(c) If \( r = 1 \), then the root test is inconclusive.

The ratio test works the same way: if the associated positive series has \( r > 1 \), then the original series diverges.
Alternating Series

An alternating series is a series whose terms alternate between positive and negative:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots
\]

Usually an alternating series will have something like a \((-1)^n\) in the formula, since \((-1)^n\) alternates between -1 and +1. (The above series uses \((-1)^{n-1}\), which has the advantage of making the \(n = 1\) term positive.)

The series above is known as the alternating harmonic series. Clearly this series does not converge absolutely. The surprising thing is that it does converge:

Because the series is alternating, the partial sums jump back and forth (left, then right, then left, and so on). Further, each term is smaller than the last, so each jump is shorter than the previous jump. Since the size of the jumps goes to zero, the series eventually narrows in on a limit (indicated by the vertical line).

THE ALTERNATING SERIES TEST

Let \(\sum a_n\) be a series. Suppose that:

1. The terms \(a_n\) alternate between positive and negative.
2. Each term is smaller than the last: \(|a_{n+1}| \leq |a_n|\).
3. The terms go to zero: \(\lim_{n \to \infty} a_n = 0\).

Then the series \(\sum a_n\) converges.
EXAMPLE 7  Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges or diverges.

SOLUTION  Note that $\sum_{n=2}^{\infty} \frac{1}{\ln n} = \infty$, so the given series does not converge absolutely.

But does it converge? The $(-1)^n$ makes it an alternating series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$$

Each term is smaller than the last, and the terms go to zero (since $1/\ln n \to 0$ as $n \to \infty$). Therefore, the series converges by the Alternating Series Test.

If a series converges by the Alternating Series Test, then the partial sums must jump back and forth across the limit. For example, let $s$ be the sum of the alternating harmonic series:

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

Then $s$ is greater than the fourth partial sum:

$$s \geq 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

but less than the fifth partial sum:

$$s \leq 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

In particular, the fourth partial sum approximates $s$ to within $1/5$.

The difference between the sum of a series and the $n$th partial sum is called the remainder:

$$R_n = s - s_n$$

The remainder measures how accurate it is to approximate the sum of the series by the $n$th partial sum. For the alternating harmonic series, we have shown that $|R_4| \leq 1/5$.

THE ALTERNATING SERIES ESTIMATION THEOREM

Suppose $\sum a_n$ is a series that converges by the Alternating Series Test.

Let $s$ be the sum of the series, let $s_n$ be the $n$th partial sum, and let $R_n = s - s_n$.

Then:

$$|R_n| \leq |a_{n+1}|$$
EXAMPLE 8  Find the maximum error in the following approximation:

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \approx \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}
\]

SOLUTION  By the Alternating Series Estimation Theorem:

\[
|R_4| \leq \left| \frac{1}{5!} \right| = \frac{1}{5!} = \frac{1}{120} \approx 0.008
\]

EXAMPLE 9  Estimate the sum of the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \) to within 0.01.

SOLUTION  Note that this series converges by the Alternating Series Test. Here are the first few terms:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125} - \cdots
\]

The fifth term is smaller than 0.01. Therefore, the sum of the series is within 0.01 of the fourth partial sum:

\[
1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} \approx 0.896
\]

EXAMPLE 10  How many terms of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) would we need to add to be within 0.01 of the actual sum?

SOLUTION  By the Alternating Series Estimation Theorem:

\[
|R_n| \leq \frac{1}{\sqrt{n} + 1}
\]

If we want \( |R_n| \) to be less than 0.01, we would need:

\[
\frac{1}{\sqrt{n} + 1} \leq 0.01 \quad \text{so} \quad \sqrt{n} + 1 \geq 100 \quad \text{so} \quad n + 1 \geq 10,000
\]

and therefore \( n \) would have to be at least 9999.

Note that the series in the previous example does not converge absolutely, because \( 1/\sqrt{n} \) does not go to zero quickly enough. This is related to why you must add together so many terms before the series gets within 0.01 of its limit. In general, a series that converges but does not converge absolutely approaches its limit very slowly.
EXERCISES

1–6 Determine whether the given alternating series converges or diverges.

1. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \)
2. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}} \)
3. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{3^{1/n}} \)
4. \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 1}} \)
5. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1 + \ln n} \)
6. \( \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + e^{-n}} \)

7–20 Determine whether the given series converges. If it does converge, determine whether it converges absolutely.

7. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \)
8. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sqrt{n}} \)
9. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \)
10. \( \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln(n^2)} \)
11. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n + 3} \)
12. \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1} \)
13. \( \sum_{n=1}^{\infty} \frac{\cos n}{n \sqrt{n}} \)
14. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + \cos^2 n} \)
15. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\tan^{-1} n} \)
16. \( \sum_{n=0}^{\infty} \frac{\sin^3(n)}{n!} \)
17. \( \sum_{n=2}^{\infty} \frac{\sin(\sqrt{n})}{n(\ln n)^2} \)
18. \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \)
19. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^6 + 4}} \)
20. \( \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n} \)

21–26 Use the root test to determine whether the given series converges or diverges.

21. \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \)
22. \( \sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n^{3^n}} \)
23. \( \sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n \sqrt{n + 1}} \)
24. \( \sum_{n=1}^{\infty} \frac{(3e)^n}{(-2)^n 5^n \sqrt{n}} \)
25. \( \sum_{n=1}^{\infty} \frac{(-3)^n n^3}{2^{2n+1}} \)
26. \( \sum_{n=2}^{\infty} \frac{(-1)^n n!}{3^n \ln n} \)

27. Estimate the error in the following approximation:
   \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \approx \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} \]

28. Estimate the error in the following approximation:
   \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \approx -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} \]

29. Estimate the error in using \( s_5 \) to approximate \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \).

30. Estimate the error in the following approximation:
   \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \approx \sum_{n=1}^{99} \frac{(-1)^n}{n^3} \]

31. Estimate \( \sum_{n=1}^{\infty} \frac{n}{(-10)^n} \) to within 0.0001.

32. Estimate \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \) to within 0.01.

33. Estimate \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n n} \) to within 0.001.

34. Estimate \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \) to within 0.00001.

35. How many terms of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) would we need to add to be within 0.001 of the actual sum?

36. How many terms of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n} \) would we need to add to be within 0.1 of the actual sum?

37. Suppose that the \( n \)th partial sum of a series \( \sum_{n=1}^{\infty} a_n \) is
   \[ s_n = \frac{n}{n + 1} \]

   (a) Find the sum of the series.
   (b) Find a formula for \( a_n \).

38. Repeat the steps in #37 for \( s_n = \frac{n - 1}{2n + 1} \).