A **sequence** is an infinite list of numbers written in a definite order:

\[2, \ 4, \ 8, \ 16, \ 32, \ \ldots\]

The numbers in the list are called the **terms** of the sequence. In the above sequence, the first term is 2, the second term is 4, the third term is 8, and so on, each successive term being twice the previous term.

Suppose we wanted to figure out the 10th term in this sequence. How could we manage this? One method would be to continue the doubling process:

\[2, \ 4, \ 8, \ 16, \ 32, \ 64, \ 128, \ 256, \ 512, \ 1024, \ \ldots\]

Calculating all these numbers was quite tedious, but we have managed to figure out that the 10th term is 1024. A better method is to find a **formula** for the terms of the sequence. In this case, we might notice that the terms are just the powers of 2:

\[2^1 = 2, \ 2^2 = 4, \ 2^3 = 8, \ 2^4 = 16, \ 2^5 = 32, \ \ldots\]

In general, the \(n\)th term of this sequence is \(2^n\), so the 10th term must be \(2^{10}\), which is 1024 (according to my calculator).

**Different Ways of Writing a Sequence**

It’s often clearer when specifying a sequence to give a formula for the \(n\)th term immediately. For example, it would have been clearer to write the above sequence as:

\[2, \ 4, \ 8, \ 16, \ \ldots, \ 2^n, \ \ldots\]

Some other ways of writing this sequence include:

\[\{2, 4, 8, 16, \ldots\} \quad \{2^n\} \quad \text{or} \quad \{2^n\}_{n=1}^{\infty}\]

Sometimes we will want to talk about sequences whose terms are variables:

\[a_1, \ a_2, \ a_3, \ a_4, \ \ldots\]

The common practice is to use the same letter for all the terms in a sequence (in this case \(a\)), with subscripts to distinguish between different terms. Other ways of writing this same sequence include:

\[\{a_1, a_2, a_3, \ldots\} \quad \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}\]
Formulas for Sequences

**EXAMPLE 1** Find formulas for the following sequences:

(a) \(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\)

(b) \(2, 4, 6, 8, 10, \ldots\)

(c) \(1, 4, 9, 16, 25, \ldots\)

**SOLUTION**

(a) This is obviously the sequence \(\{1/n\}\).

(b) Usually a good way of figuring out the formula is to write each term next to the corresponding \(n\):

\[
\begin{array}{c|c|c|c|c|c|c|c}
 n & 1 & 2 & 3 & 4 & 5 & \ldots \\
 a_n & 2 & 4 & 6 & 8 & 10 & \ldots \\
\end{array}
\]

As you can see, each term is twice the corresponding \(n\), so this is the sequence \(\{2n\}\).

(c) We write each term next to the corresponding \(n\):

\[
\begin{array}{c|c|c|c|c|c|c|c}
 n & 1 & 2 & 3 & 4 & 5 & \ldots \\
 a_n & 1 & 4 & 9 & 16 & 25 & \ldots \\
\end{array}
\]

As you can see, the \(n\)th term is equal to \(n^2\), so this is the sequence \(\{n^2\}\).

There are certain sequences that you should know on sight:

**COMMON SEQUENCES**

- \(\{2^n\}: \{2, 4, 8, 16, 32, \ldots\}\)
- \(\{3^n\}: \{3, 9, 27, 81, 243, \ldots\}\)
- \(\{n^2\}: \{1, 4, 9, 16, 25, \ldots\}\)
- \(\{n^3\}: \{1, 8, 27, 64, 125, \ldots\}\)
- \(\{n!\}: \{1, 2, 6, 24, 120, \ldots\}\)
EXAMPLE 2  Find formulas for the following sequences:

(a) \(3, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \frac{243}{120}, \ldots\)

(b) \(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\)

(c) \(\sqrt{3}, 4, 3\sqrt{5}, 4\sqrt{6}, 5\sqrt{7}, \ldots\)

(d) \(16, 25, 36, 49, 64, \ldots\)

SOLUTION

(a) This is the sequence \(\left\{\frac{3^n}{n!}\right\}\).

(b) Let's compare \(a_n\) with \(n\):

\[
\begin{array}{c|cccccc}
 n & 1 & 2 & 3 & 4 & 5 & \ldots \\
 a_n & 0 & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \ldots \\
\end{array}
\]

Clearly the first term is really \(\frac{0}{1}\). The denominator of the fraction is always \(n\), and the numerator is \(n - 1\), so this is the sequence \(\left\{\frac{n-1}{n}\right\}\).

(c) Let's compare \(a_n\) with \(n\):

\[
\begin{array}{c|cccccc}
 n & 1 & 2 & 3 & 4 & 5 & \ldots \\
 a_n & \sqrt{3} & 4 & 3\sqrt{5} & 4\sqrt{6} & 5\sqrt{7} & \ldots \\
\end{array}
\]

For the latter three terms, the coefficient of the square root is \(n\), and the number inside the square root is \(n + 2\). This formula also works for the first and second terms:

\[\sqrt{3} = 1\sqrt{3} \quad 4 = 2\sqrt{4}\]

Therefore, this is the sequence \(\{n\sqrt{n+2}\}\).

(d) Each of the terms in this sequence is a perfect square. Indeed:

\[
\begin{array}{c|cccccc}
 n & 1 & 2 & 3 & 4 & 5 & \ldots \\
 a_n & 16 = 4^2 & 25 = 5^2 & 36 = 6^2 & 49 = 7^2 & 64 = 8^2 & \ldots \\
\end{array}
\]

The number being squared is always \(n + 3\), so this is the sequence \(\{(n + 3)^2\}\). ■
The Limit of a Sequence

You can take the limit of a sequence as \( \{a_n\} \) as \( n \to \infty \) in the same way that you take the limit of a function \( f(x) \) as \( x \to \infty \). The only difference is that there is one term \( a_n \) for every positive integer \( n \), while there is one value of \( f(x) \) for every real number.

CONVERGENCE AND DIVERGENCE

We say the sequence \( \{a_n\} \) **converges** if \( \lim_{n \to \infty} a_n \) is a real number.

If \( \lim_{n \to \infty} a_n \) is infinite or does not exist, the sequence **diverges**.

**EXAMPLE 3**  Does the sequence \( \{n^2\} \) converge or diverge?

**SOLUTION**  Since:

\[
\lim_{n \to \infty} n^2 = \infty
\]

the sequence diverges to \( \infty \).

**EXAMPLE 4**  Does the sequence \( \left\{ \frac{n^2 + 1}{3n^2 + 4n + 2} \right\}_{n=1}^\infty \) converge or diverge?

**SOLUTION**  We have:

\[
\lim_{n \to \infty} \frac{n^2 + 1}{3n^2 + 4n + 2} = \lim_{n \to \infty} \frac{n^2}{3n^2} = \frac{1}{3}
\]

Therefore, the sequence converges to \( \frac{1}{3} \).

**EXAMPLE 5**  Does the sequence:

\[
0, \ 1, \ 0, \ 1, \ 0, \ 1, \ ... 
\]

converge or diverge?

**SOLUTION**  This sequence oscillates between 0 and 1, and therefore has no limit as \( n \to \infty \). (It approaches neither 0 nor 1, so the limit does not exist. This is similar to the limit \( \lim_{x \to \infty} \sin x \).) The sequence therefore diverges.
EXAMPLE 6 Determine whether the sequence $a_n = \frac{\cos n}{n!}$ converges or diverges.

**SOLUTION** Recall that $-1 \leq \cos n \leq 1$ for every value of $n$, so:

$$\frac{\cos n}{n!} = \frac{\text{something between } -1 \text{ and } 1}{\text{something that goes to } \infty} \to 0 \quad \text{as } n \to \infty$$

Therefore, the sequence $\left\{\frac{\cos n}{n!}\right\}$ converges to zero.

EXAMPLE 7 Determine whether the sequence $\left\{\tan^{-1}(n)\right\}$ converges or diverges.

**SOLUTION** Recall that inverse tangent is the function that converts slopes to angles:

$$\left(\begin{array}{c}
\text{slope of} \\
\text{a line}
\end{array}\right) \xrightarrow{\tan^{-1}} \left(\begin{array}{c}
\text{angle the line makes} \\
\text{with the } x\text{-axis}
\end{array}\right)$$

For example:

- $\tan^{-1}(0) = 0$
- $\tan^{-1}(1) = \frac{\pi}{4}$
- $\tan^{-1}(\infty) = \frac{\pi}{2}$

Therefore:

$$\lim_{n \to \infty} \tan^{-1}(n) = \frac{\pi}{2}$$

so the sequence converges to $\pi/2$.

For more on limits of sequences, see the notes on limits at infinity.
EXERCISES

1–10 ■ Find a formula for the general term $a_n$ of the sequence, assuming the pattern of the first few terms continues.

1. \[ \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\} \]

2. \[ \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots \right\} \]

3. \[ \left\{ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots \right\} \]

4. \[ \left\{ \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \ldots \right\} \]

5. \[ \left\{ 1, \frac{2}{4}, \frac{6}{9}, \frac{24}{16}, \frac{120}{25}, \ldots \right\} \]

6. \[ \left\{ \frac{1!}{1!}, \frac{2!}{3!}, \frac{3!}{5!}, \frac{4!}{7!}, \ldots \right\} \]

7. \[ \left\{ \frac{2}{5}, \frac{4}{25}, \frac{8}{125}, \frac{16}{625}, \ldots \right\} \]

8. \{6, 12, 24, 48, 96, \ldots \}

9. \[ \left\{ \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \frac{80}{81}, \ldots \right\} \]

10. \[ \left\{ 10, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots \right\} \]

11–18 ■ Determine whether the given sequence converges or diverges. If it converges, find the limit.

11. $a_n = \frac{n^2 - 1}{n^2 + 1}$

12. $a_n = (-1)^n$

13. $a_n = \cos\left(\frac{n\pi}{2}\right)$

14. $\{\cos(2\pi n)\}$

15. \[ \left\{ \frac{n \tan^{-1} n}{2n + 1} \right\}_{n=1}^{\infty} \]

16. $a_n = \frac{\sin n}{n}$

17. \[ \left\{ \frac{\cos^2 n}{2^n} \right\} \]

18. $a_n = \sqrt[4]{4}$