Answers (Section 8.4 Summary, Part B)
C = converges,  D = diverges,  CA = converges absolutely, CNA = converges, but not absolutely

22. D (r = \infty)  23. D (r = e)  24. CA (r = 3e/10)  25. CA (r = 3/4)  26. D (r = \infty)
27. \frac{1}{\ln 6} \approx 0.56  28. \frac{1}{49} \approx 0.02  29. \frac{1}{720^2} \approx 0.000002  30. 0.000001
31. -0.0826  32. -0.95  33. 0.783  34. -0.77611  35. 31 terms  36. 22,026 terms
37. (a) 1  (b) a_n = \frac{1}{n(n + 1)}  38. (a) 1/2  (b) a_n = \frac{3}{(2n + 1)(2n - 1)}
Solutions
(Section 8.4 Summary, Part B)

1. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots. \] This series is alternating, each term is smaller than the last, and the terms go to zero. Therefore, this series \textbf{converges} by the Alternating Series Test.

2. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}} = -1 + \frac{1}{2^{3/4}} - \frac{1}{3^{3/4}} + \frac{1}{4^{3/4}} - \cdots. \] This series is alternating, each term is smaller than the last, and the terms go to zero. Therefore, this series \textbf{converges} by the Alternating Series Test.

3. Since \( \lim_{n \to \infty} 3^{1/n} = 3^0 = 1 \), the terms of this series don't go to zero. Therefore, this series \textbf{diverges} by the Divergence Test.

4. \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{17}} - \cdots. \] This series is alternating, each term is smaller than the last, and the terms go to zero. Therefore, this series \textbf{converges} by the Alternating Series Test.

5. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1 + \ln n} = 1 - \frac{1}{1 + \ln 2} + \frac{1}{1 + \ln 3} - \frac{1}{1 + \ln 4} + \cdots. \] This series is alternating, each term is smaller than the last, and the terms go to zero. Therefore, this series \textbf{converges} by the Alternating Series Test.

6. Since \( \lim_{n \to \infty} 1 + e^{-n} = 1 + 0 = 1 \), the terms of this series don't got zero. Therefore, this series \textbf{diverges} by the Divergence Test.

7. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges.} \] Therefore, the given series \textbf{converges absolutely}.

8. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sqrt{n}} \text{ converges (p-series with } p = 3/2). \] Therefore, the given series \textbf{converges absolutely}. 


9. This series **converges** by the Alternating Series Test. However, 
\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}
\]
\((p\text{-series with } p = 1/3), \text{ so the series does not converge absolutely.} \)

10. This series **converges** by the Alternating Series Test. However, 
\[
\sum_{n=2}^{\infty} \frac{1}{2 \ln n} \text{ diverges}, \text{ so the series does not converge absolutely.}
\]

11. Since \( \lim_{n \to \infty} \frac{n}{2n + 3} = \frac{1}{2} \), the terms of this series don't go to zero. Therefore, this series **diverges** by the Divergence Test.

12. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} = -\frac{1}{2} + \frac{2}{5} - \frac{3}{10} + \frac{4}{17} - \frac{5}{26} + \cdots \). This series is alternating, each term is smaller than the last, and the terms go to zero since \( \lim_{n \to \infty} \frac{n}{n^2 + 1} = 0 \). Therefore, this series **converges** by the Alternating Series Test. However, 
\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2 + 1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \text{ diverges since } \frac{n}{n^2 + 1} \sim \frac{1}{n}, \text{ so the series does not converge absolutely.}
\]

13. \( \sum_{n=1}^{\infty} \left| \cos n \right| = \sum_{n=1}^{\infty} \frac{\left| \cos n \right|}{n^{1/2}} \). But \( \left| \cos n \right| \leq 1 \), so \( \frac{\left| \cos n \right|}{n^{1/2}} \leq \frac{1}{n^{1/2}} \). Since \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \) converges \((p\text{-series with } p = 3/2), \text{ the smaller series } \sum_{n=1}^{\infty} \left| \frac{\cos n}{n^{1/2}} \right| \text{ converges by the Comparison Test, and therefore the given series **converges absolutely.} \)

14. Since \( 0 \leq \cos^2 n \leq 1 \), the denominator is always between 1 and 2. This means that the terms of the series don't go to zero, so the series **diverges** by the Divergence Test.

15. Since \( \lim_{n \to \infty} \tan^{-1}(n) = \pi/2 \), the terms of this series don't go to zero. Therefore, this series **diverges** by the Divergence Test.

16. \( \sum_{n=0}^{\infty} \left| \frac{\sin^3(n)}{n!} \right| = \sum_{n=0}^{\infty} \frac{|\sin n|^3}{n!} \). Since \( 0 \leq |\sin n| \leq 1 \), we see that \( 0 \leq |\sin n|^3 \leq 1 \), so that \( \frac{|\sin n|^3}{n!} \leq \frac{1}{n!} \). Since \( \sum_{n=0}^{\infty} \frac{1}{n!} \) converges, the smaller series \( \sum_{n=0}^{\infty} \left| \frac{\sin^3(n)}{n!} \right| \) converges by the Comparison Test, and therefore the given series **converges absolutely.**
17. \[ \sum_{n=2}^{\infty} \frac{\sin\left(\sqrt{n}\right)}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{\sin\sqrt{n}}{n(\ln n)^2} . \] Since \( 0 \leq \sin\sqrt{n} \leq 1 \), we see that \( \frac{\sin\sqrt{n}}{n(\ln n)^2} \leq \frac{1}{n(\ln n)^2} \). But \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges by the integral test:

\[
\begin{align*}
    u &= \ln x \\
    du &= \frac{1}{x} \, dx \\
    \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx &= \int_{\ln 2}^{\infty} \frac{1}{u^2} \, du \quad (\text{converges since } 2 > 1)
\end{align*}
\]

Thus the smaller series \( \sum_{n=2}^{\infty} \frac{\sin(\sqrt{n})}{n(\ln n)^2} \) converges by the Comparison Test, so the given series converges absolutely.

18. \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} = \frac{1}{2 \ln 2} - \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} - \frac{1}{5 \ln 5} + \cdots . \] This series is alternating, each term is smaller than the last, and the terms go to zero. Therefore, this series converges by the Alternating Series Test. However, the associated positive series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) diverges by the Integral Test:

\[
\begin{align*}
    u &= \ln x \\
    du &= \frac{1}{x} \, dx \\
    \int_{2}^{\infty} \frac{1}{x \ln x} \, dx &= \int_{\ln 2}^{\infty} \frac{1}{u} \, du \quad (\text{diverges since } 1 \leq 1)
\end{align*}
\]

Therefore, the given series does not converge absolutely.

19. \[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^6 + 4}} = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6 + 4}} \] converges since \( \frac{n}{\sqrt{n^6 + 4}} \sim \frac{n}{n^3} = \frac{1}{n^2} \). Therefore, the given series converges absolutely.

20. Since \( \lim_{n \to \infty} \frac{\sqrt{n}}{\ln n} = \infty \), the terms of this series don't go to zero. Therefore, this series diverges by the Divergence Test.

21. \[ \left| \frac{(-2)^n}{n!} \right| = \frac{2^n}{n!} , \text{ which has } r = \frac{2}{\infty} = 0 . \] Therefore, this series converges absolutely.

22. \[ \left| \frac{(-1)^n n^n}{n 3^n} \right| = \frac{n^n}{n 3^n} , \text{ which has } r = \frac{\infty}{1 \cdot 3} = \infty . \] Therefore, this series diverges.
23. \[ \lim_{n \to \infty} \frac{(-1)^n e^n}{n \sqrt{n + 1}} = \frac{e^n}{n \sqrt{n + 1}}, \text{ which has } r = \frac{e}{1} = e > 1. \] Therefore, this series diverges.

24. \[ \lim_{n \to \infty} \frac{(3e)^n}{(-2)^n 5^{\sqrt{n}/n}} = \frac{(3e)^n}{2^n 5^{\sqrt{n}/n}}, \text{ which has } r = \frac{3e}{2 \cdot 5 \cdot 1} = \frac{3e}{10} < 1. \] Therefore, this series converges absolutely.

25. \[ \lim_{n \to \infty} \frac{(-3)^n n^n}{2^{2n+1}} = \frac{3^n n^3}{2^{2n+1}}, \text{ which has } r = \frac{3 \cdot 1}{2^2} = \frac{3}{4} < 1. \] Therefore, this series converges absolutely.

26. \[ \lim_{n \to \infty} \frac{(-1)^n n!}{3^n \ln n} = \frac{n!}{3^n \ln n}, \text{ which has } r = \frac{\infty}{3^3 \cdot 1} = \infty. \] Therefore, this series diverges.

27. By the Alternating Series Estimation Theorem, \( |R_5| \leq \frac{1}{\ln 6} \approx 0.56. \)

28. By the Alternating Series Estimation Theorem, \( |R_6| \leq \frac{1}{49} \approx 0.02. \)

29. By the Alternating Series Estimation Theorem, \( |R_5| \leq \frac{1}{(6!)^2} = \frac{1}{720^2}, \) which is about 0.000002.

30. By the Alternating Series Estimation Theorem, \( |R_{100}| \leq \frac{1}{1,000,000} = 0.000001. \)

31. Here are the first few terms of the series:

\[
\sum_{n=1}^{\infty} \frac{n}{(-10)^n} = \frac{-1}{10} + \frac{2}{100} - \frac{3}{1000} + \frac{4}{10,000} - \frac{5}{100,000} + \cdots \\
= -0.1 + 0.02 - 0.003 + 0.0004 - 0.00005 + \cdots
\]

By the Alternating Series Estimation Theorem, the fourth partial sum is within 0.00005, and hence within 0.0001:

\[-0.1 + 0.02 - 0.003 + 0.0004 = -0.0826\]
32. We must be accurate to within $0.01 = 1/100$. Here are the first few terms of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -1 + \frac{1}{16} - \frac{1}{81} + \frac{1}{256} - \cdots$$

Since $1/256$ is smaller than $1/100$, the third partial sum is within $0.01$:

$$-1 + \frac{1}{16} - \frac{1}{81} \approx -0.95$$

33. We must be accurate to within $0.001 = 1/1000$. Here are the first few terms of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n} = 1 - \frac{1}{4} + \frac{1}{27} - \frac{1}{256} + \frac{1}{3125} - \cdots$$

Since $1/3125$ is smaller than $1/1000$, the fourth partial sum is correct to within $0.001$:

$$1 - \frac{1}{4} + \frac{1}{27} - \frac{1}{256} = 0.783$$

34. We must be accurate to within $0.00001 = 1/100,000$. Here are the first few terms of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} = -1 + \frac{1}{2^2} - \frac{1}{6^2} + \frac{1}{24^2} - \frac{1}{120^2} + \frac{1}{720^2} + \cdots$$

The sixth term $1/720^2$ ought to be smaller than $1/100,000$. Therefore, the fifth partial sum is correct to within $0.00001$:

$$-1 + \frac{1}{2^2} - \frac{1}{6^2} + \frac{1}{24^2} - \frac{1}{120^2} = -0.77611$$

35. We must be accurate to within $0.001 = 1/1000$, so we are looking for the first term that's smaller than $1/1000$:

$$\frac{1}{n^2} \leq \frac{1}{1000}$$

Then $n^2$ must be bigger than $1000$, so $n$ must be bigger than $\sqrt{1000} \approx 31.6$. We conclude that the 32nd term of the series is smaller than $1/1000$, so we must add up the first 31 terms.
36. We must be accurate to within $0.1 = 1/10$, so we are looking for the first term that's smaller than $1/10$:
\[
\frac{1}{\ln n} \leq \frac{1}{10}
\]
Then $\ln n$ must be bigger than 10, so $n$ must be bigger than $e^{10} \approx 22,026.4$. We conclude that the $n = 22,027$ term of the series is smaller than $1/10$ so we must add up the first [22,026 terms].

Apparently this series converges very slowly.

37.
(a) The sum of the series is $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n}{n + 1} = [1]$.
(b) Since $s_n = s_{n-1} + a_n$, we can find $a_n$ by subtracting two consecutive partial sums:
\[
a_n = s_n - s_{n-1} = \frac{n}{n + 1} - \frac{n - 1}{n} = \frac{n^2 - (n - 1)(n + 1)}{n(n + 1)} = \frac{n^2 - (n^2 - 1)}{n(n + 1)} = \frac{1}{n(n + 1)}
\]

38.
(a) The sum of the series is $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n - 1}{2n + 1} = [1/2]$.
(b) Since $s_n = s_{n-1} + a_n$, we can find $a_n$ by subtracting two consecutive partial sums:
\[
a_n = s_n - s_{n-1} = \frac{n - 1}{2n + 1} - \frac{n - 2}{2n - 1} = \frac{(n - 1)(2n - 1) - (n - 2)(2n + 1)}{(2n + 1)(2n - 1)} = \frac{(2n^2 - 3n + 1) - (2n^2 - 3n - 2)}{(2n + 1)(2n - 1)} = \frac{3}{(2n + 1)(2n - 1)}
\]