Making contractions continuous: a problem related to the Kneser-Poulsen conjecture

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Abstract

We show that there exist two configurations of points \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) in \( \mathbb{E}^n \) such that \( q \) is a contraction of \( p \) and there is not a continuous contraction from \( p \) to \( q \) in \( \mathbb{E}^{2n-1} \). It is known [BC02] that there is always a continuous contraction in \( \mathbb{E}^{2n} \).

1 Introduction

Let \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) be configurations of \( N \) points where each \( p_i \in \mathbb{E}^n \) and each \( q_i \in \mathbb{E}^n \). If \( |p_i - p_j| \geq |q_i - q_j| \) for all \( i \) and \( j \), then we say that \( q \) is a contraction of \( p \) in \( \mathbb{E}^n \) (and \( p \) is an expansion of \( q \)). If there exists a continuous motion \( p(t) = (p_1(t), \ldots, p_N(t)) \) with \( p_i(t) \in \mathbb{E}^m \) for all \( 0 \leq t \leq 1 \) and \( 0 \leq i \leq N \) such that \( p(0) = p \) and \( q(0) = q \) and \( |p_i(t) - p_j(t)| \) is monotone decreasing for all \( i \) and \( j \), then we say that \( p(t) \) is a continuous contraction from \( p \) to \( q \) in \( \mathbb{E}^m \).

In [BC02], Bezdek and Connelly prove the following lemma (see also [CP91] and [Gr87]):

Lemma 1. If \( q \) is a contraction of \( p \) in \( \mathbb{E}^n \) then there exists a continuous contraction from \( p \) to \( q \) in dimension \( \mathbb{E}^{2n} \).

Bezdek and Connelly [BC02] use the above lemma to prove the Kneser-Poulsen conjecture in \( \mathbb{E}^2 \). The Kneser-Poulsen conjecture was independently conjectured by Poulsen in 1954 [Po54] and by Kneser in 1955 [Kn55]:

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Conjecture. (Kneser and Poulsen) If $q$ is a contraction of $p$ in $\mathbb{E}^n$ and $B(p_i)$ and $B(q_i)$ are balls of radius $r$ centered at $p_i$ and $q_i$, respectively, then

$$\text{Vol} \left( \bigcup B(p_i) \right) \leq \text{Vol} \left( \bigcup B(q_i) \right)$$

There is an analogous conjecture for intersections. Bezdek and Connelly [BC02] prove that

**Theorem 1.** If $q$ is a contraction of $p$ in $\mathbb{E}^n$ and there exists a continuous contraction from $p$ to $q$ in $\mathbb{E}^{n+2}$, then

$$\text{Vol} \left( \bigcup B(p_i) \right) \leq \text{Vol} \left( \bigcup B(q_i) \right)$$

This theorem along with Lemma 1 allows them to prove the Kneser-Poulsen conjecture in dimension 2.

In this paper, we will prove that Lemma 1 is sharp; in particular we will prove:

**Theorem 2.** There exist configurations $p$ and $q$ in $\mathbb{E}^n$ such that $q$ is a contraction of $p$ and there is no continuous contraction from $p$ to $q$ in $\mathbb{E}^{2n-1}$.

Thus, it is not possible to use Theorem 1 to prove the Kneser-Poulsen conjecture in dimensions greater than 2. This leads to the following open questions:

**Question 1.** Is there a counterexample to the Kneser-Poulsen conjecture using the configuration from Theorem 2?

The configuration in Theorem 2 uses $(n + 1)^2$ points. Clearly, $2n$ points can always continuously contract in $\mathbb{E}^{2n-1}$.

**Question 2.** Are there configurations with fewer than $(n + 1)^2$ points satisfying Theorem 2?

## 2 Second Order Rigidity

To prove Theorem 2, we will show that a certain bar framework is rigid in $\mathbb{E}^{2n-1}$. First, we will need some definitions and theorems about bar frameworks. The definitions and theorems in this section come from [CW96].
Definition 1. A bar framework \( G(p) \) is a configuration \( p = (p_1, \ldots, p_n) \) combined with a graph \( G \) where vertices in the graph correspond to points in the configuration. Edges in the graph are called bars.

A flex of a bar framework \( G(p) \) is a continuous motion \( p(t) = (p_1(t), \ldots, p_N(t)) \) where \( p(0) = p \) and adjacent vertices in the graph are constrained to remain the same distance apart; that is if \( \{i, j\} \in E(G) \) then \( |p_i(t) - p_j(t)| \) is constant for \( t > 0 \). A trivial flex is a flex where \( p(t) \) is congruent to \( p \) for all \( t \). A bar framework is rigid if every flex is trivial.

If there is a flex \( p(t) \) of a bar framework, for every bar \( \{i, j\} \) in a bar framework, the equation \( |p_j(t) - p_i(t)|^2 = L_{ij} \) must be satisfied (where \( L_{ij} \) is a constant). Differentiating this equation twice, we get requirements on the first and second order flexes \( p' \) and \( p'' \). This gives us the definition of a second order flex for a bar framework:

Definition 2. A second order flex \( (p', p'') \) for a bar framework \( G(p) \) is a solution to the following constraints, where \( p' \) and \( p'' \) are configurations in \( \mathbb{E}^n \) (each regarded as an associated pair of vectors \( p'_i \) and \( p''_i \) to each point \( p_i \)):

\[
(p_i - p_j) \cdot (p'_i - p'_j) = 0 \tag{1}
\]

and

\[
|p'_i - p'_j|^2 + (p_i - p_j) \cdot (p''_i - p''_j) = 0
\]

A bar framework is second-order rigid if all second-order flexes \( (p', p'') \) have \( p' \) as a trivial first-order flex.

Definition 3. A stress on a bar framework \( G(p) \) is an assignment of real numbers \( \omega_{ij} = \omega_{ji} \) to the edges of \( G \), where \( \omega = (\ldots, \omega_{ij}, \ldots) \in \mathbb{R}^e \), and \( e \) is the number of edges. A stress \( \omega \) on a bar framework is an equilibrium stress if the following equilibrium condition holds at each vertex \( i \):

\[
\sum_j \omega_{ij}(p_j - p_i) = 0
\]

where the sum is taken over all \( j \) with \( \{i, j\} \in E \).
Definition 4. If $A$ is an $m \times n$ matrix and $B$ is a $q \times p$ matrix, then the Kronecker product $A \otimes B$ is the $mq \times np$ block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

Definition 5. Let $G(p)$ be a bar framework with $e$ edges and $v$ vertices, and let $\omega = (\ldots, \omega_{ij}, \ldots) \in \mathbb{R}^e$ be a stress for $G(p)$. The reduced stress matrix $\Omega$ is the $v \times v$ symmetric matrix with entries

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{if } i \neq j \\ \sum_k \omega_{ik} & \text{if } i = j \end{cases}$$

The stress matrix $\Omega$ is the matrix $\Omega \otimes I^n$, where $I^n$ is the $n \times n$ identity matrix.

We will use the following theorems to prove the rigidity of a specific bar framework. These theorems come from [CW96].

Theorem 3. If a bar framework framework $G(p)$ is second-order rigid, then it is rigid.

Theorem 4. A first-order flex $p'$ of a bar framework $G(p)$ extends to a second-order flex if and only if for all equilibrium stresses $\omega$ for $G(p)$, with stress matrix $\Omega$,

$$(p')^T \Omega p' = 0$$

3 The Simplex with Flaps

We will call the bar framework in question the $n$-simplex with flaps. We will construct it in $\mathbb{E}^n$. This is the same configuration that is used in [BC02] to show that a contraction in $\mathbb{E}^n$ does not necessarily have a continuous contraction in $\mathbb{E}^{n+1}$.

We start with a regular $n$-simplex, with $n + 1$ vertices and $\binom{n+1}{2}$ edges. The “flaps” extend perpendicularly from each facet of the simplex: take the vertices of one facet; translate copies of these vertices a distance $L$ in the direction perpendicular to the facet and away from the inside of
the simplex; now, add edges connecting all \( \binom{n}{2} \) of these vertices, and add edges connecting each of these vertices to each vertex in the original facet. Thus, the flaps are completely rigid.

Each vertex of the original simplex is associated with \( n \) vertices from the flaps (the \( n \) vertices that were created by translating this vertex). Add edges connecting these \( n \) vertices to each other.

We call the resulting bar framework the \( n \)-simplex with flaps. Figure 1 shows the triangle with flaps.

This bar framework has two configurations in dimension \( n \) that preserve the edge lengths of the bars. One configuration is the configuration that we have created. The other configuration is the \( n \)-simplex with the flaps attached to the other side of each facet. Note that this is a contraction of the vertices of the framework. We will show that the framework is rigid in \( \mathbb{E}^{2n-1} \), and thus there is no continuous contraction between the two configurations in \( \mathbb{E}^{2n-1} \).

The plan is to use Theorem 4 to show that the \( n \)-simplex with flaps is second order rigid in \( \mathbb{E}^{2n-1} \). It will be helpful to note some facts about the possible infinitesimal motions \( p' \) and the possible equilibrium stresses \( \omega \) of the \( n \)-simplex with flaps.
Figure 2: The orthogonal circles in Lemma 3.

First, note that the initial simplex must be completely rigid, so that any infinitesimal flex of the simplex must extend to a congruence. Thus, we can assume that the infinitesimal flex is 0 on the simplex.

**Lemma 2.** Let $p'$ be an infinitesimal flex of the $n$-simplex with flaps. If $p_i$ and $p_j$ are vertices in the same flap, then $p_i' = p_j'$.

**Proof.** A flap of the $n$-simplex with flaps is rigid, so any infinitesimal flex must extend to a congruence.

Since the infinitesimal flex is 0 on the simplex, and since all vertices in a flap have the same infinitesimal flex, we can label the components of the flex based on the faces of the simplex. Let $1, \ldots, n + 1$ be the faces of the simplex, and let $a'_1, \ldots, a'_{n+1}$ be the flexes on the corresponding flaps.

### 4 Rigidity of the Tetrahedron with Flaps

We will prove that the tetrahedron with flaps (the 3-simplex with flaps) is rigid in $\mathbb{E}^5$. We will need the following lemma:

**Lemma 3.** Let $|a| > 1$. Consider a circle $C_1$ with center $(ax, ay)$ and radius $\sqrt{a^2 - 1} |(x, y)| = \sqrt{a^2 - 1} \sqrt{x^2 + y^2}$. This circle is orthogonal to the circle $C_2$ centered at the origin with radius $|(x, y)| = \sqrt{x^2 + y^2}$.

**Proof.** There is exactly one circle centered at $(ax, ay)$ that is orthogonal to $C_2$. Let $R$ be the radius of this circle; we will show that $R = \sqrt{a^2 - 1} \sqrt{x^2 + y^2}$. 
Figure 3: This is the bar framework consisting of the tetrahedron (which is colored grey) and two vertices, both adjacent to the vertex $p_1$ in the tetrahedron with flaps. There is exactly one stress on this bar framework.

Since the two circles are orthogonal, the two radii shown in Figure 2 are perpendicular. Then, by the Pythagorean Theorem $R^2 + (x^2 + y^2) = a^2(x^2 + y^2)$. Thus, $R = \sqrt{a^2 - 1} \sqrt{x^2 + y^2}$. \hfill \Box

**Theorem 5.** The tetrahedron with flaps is second order rigid in $\mathbb{E}^5$.

*Proof.* By Theorem 4, we need to show that for all non-trivial infinitesimal flexes $p'$ there exists a stress $\omega$ with stress matrix $\Omega$ such that $(p')^T \Omega p' \neq 0$. We will show this by contradiction: we will suppose that we have a $p'$ such that for all stresses, $(p')^T \Omega p' = 0$, and we will derive a contradiction. In particular, we will find some stresses for the tetrahedron with flaps, which will give us some equations $(p')^T \Omega p' = 0$, and we will show that these equations must be inconsistent.

We can assume that $p'$ is zero on the tetrahedron.

The stresses that we will consider are the stresses that come from the following bar framework (shown in Figure 3): the vertices are the vertices of the tetrahedron and two of the vertices adjacent to the same vertex of the tetrahedron in the graph of the tetrahedron with flaps, and the bars are all of the bars from the tetrahedron with flaps that include these vertices.

This framework has exactly one stress $\omega$, and by symmetry the stress satisfies $\omega_{52} = \omega_{64}$, $\omega_{53} = \omega_{63}$, and $\omega_{51} = \omega_{61}$. Let $\Omega$ represent the stress matrix for this stress. We consider the restriction of $p'$ to this bar framework. Since $p'$ is a second order flex, we have that $(p')^T \Omega p' = 0$. Thus,

$$\omega_{56} |p'_5 - p'_6|^2 + (\omega_{52} + \omega_{53} + \omega_{51}) |p'_5|^2 + (\omega_{63} + \omega_{64} + \omega_{61}) |p'_6|^2 = 0$$
Recall that $p'_1$, $p'_2$, $p'_3$, and $p'_4$ are all zero. The flex on $p_5$ and $p_6$ must be of the form $p'_5 = (0, 0, 0, x_5, y_5)$ and $p'_6 = (0, 0, 0, x_6, y_6)$. Let $\alpha = \omega_{52} + \omega_{53} + \omega_{51} = \omega_{64} + \omega_{63} + \omega_{61}$ (these are equal since $\omega_{52} = \omega_{64}$, $\omega_{53} = \omega_{63}$, and $\omega_{51} = \omega_{61}$). Then, the equation becomes

$$\omega_{56} ((x_5 - x_6)^2 + (y_5 - y_6)^2) + \alpha(x_5^2 + y_5^2 + x_6^2 + y_6^2) = 0$$

This implies

$$(\omega_{56} + \alpha)(x_5^2 + y_5^2 + x_6^2 + y_6^2) - 2\omega_{56}(x_5 x_6 + y_5 y_6) = 0$$

Completing the square for $x_5$ and $y_5$ and letting $k = \frac{\omega_{56}}{\omega_{56} + \alpha}$ we get

$$(x_5 - kx_6)^2 + (y_5 - ky_6)^2 = (k^2 - 1)(x_6^2 + y_6^2)$$

If we consider $p'_5$ and $p'_6$ to be points in $\mathbb{R}^2$ (the points $(x_5, y_5)$ and $(x_6, y_6)$), then $p'_5$ lies on a circle centered at $kp'_6$ with radius $\sqrt{k^2 - 1|p'_6|}$.

Recall that the stress we are currently considering comes just from the framework in Figure 3. We can also consider the stress on the analogous frameworks containing the tetrahedron and two vertices adjacent to one of vertices $p_2$, $p_3$, or $p_4$. Each of these stresses will give us an analogous equation, and we’ll get that $p'_i$ must lie on a circle centered at $kp'_j$ with radius $\sqrt{k^2 - 1|p'_j|}$.

We have been labeling the flexes based on the vertices, but this is not convenient. We should label them based on the faces of the tetrahedron. We label the faces 1, 2, 3, 4 and then we get the flexes $a'_1$, $a'_2$, $a'_3$, $a'_4$ with $a'_i = (0, 0, 0, x_i, y_i)$ being the flex for ever vertex coming from face $i$.

If we consider these points to be in $\mathbb{R}^2$, then we have 4 points in $\mathbb{R}^2$, $a'_1 = (x_1, y_1)$, $a'_2 = (x_2, y_2)$, $a'_3 = (x_3, y_3)$, $a'_4 = (x_4, y_4)$, such that any three of these points lie on a circle with radius $\sqrt{k^2 - 1|a'_i|}$ and center $ka'_i$, where $a'_i$ is the point not on this circle. By Lemma 3, the circle is orthogonal to the circle centered at the origin with radius $|a'_i|$.

Note that the four points are distinct. If $a'_i = a'_j$, then $a'_i$ would lie on a circle orthogonal to the circle centered at the origin with radius $|a'_i|$.

Now we have four points such that the circle through any three of the points (with $a'_i$ being the point not on the circle) is centered at $ka'_i$ and is orthogonal to the circle centered at the origin with radius $|a'_i|$. We want to show that this produces a contradiction.
Note that for all \(i\), \(a'_i\) is either inside or outside the circle centered at \(ka'_i\). If \(k > 0\), then \(a'_i\) is inside the circle centered at \(ka'_i\), because \(a'_i\) is on the orthogonal circle centered at the origin. If \(k < 0\), then \(a'_i\) is clearly outside of the circle centered at \(ka'_i\).

**Case 1.** For all \(i\), \(a'_i\) is inside the circle.

Note that every intersection of two of the circles is one of the \(a'_i\), because there are two \(a'_i\) that are on both circles and the \(a'_i\) must be distinct.

Consider the union of these four circles and their interiors. The boundary of this union can contain none of the \(a'_i\), since every \(a'_i\) is contained in the interior of one circle. But the \(a'_i\) are the intersection points of the circles, so at least one \(a'_i\) must be on the boundary. Thus, we have a contradiction.

**Case 2.** For all \(i\), \(a'_i\) is outside the circle.

Consider the intersection of these four circles and their interiors. Since any three circles have a nonempty intersection, the intersection of all four circles is nonempty by Helly’s Theorem. The boundary of this intersection can contain none of the \(a'_i\), since every \(a'_i\) lies outside of one of the circles. But, the \(a'_i\) are the intersection points of the circles, so at least one \(a'_i\) must be on the boundary. Thus, we have a contradiction.

Thus, the tetrahedron with flaps is second order rigid in \(E^5\). \(\square\)

### 5 Rigidity of the \(n\)-Simplex with Flaps

The proof from the previous section can be generalized to higher dimensions, where we have an \(n\)-simplex with flaps.

**Theorem 6.** The \(n\)-simplex with flaps is second-order rigid in \(E^{2n-1}\).

**Proof.** This proof will be fairly similar to the proof of Theorem 5. As in the proof of Theorem 5, we will assume that we have a flex \(p'\) such that \((p')^T \Omega p' = 0\) for all stresses. We can label the flex at each vertex by the facets of the \(n\)-simplex. We will call these flexes \(a'_1, \ldots, a'_{n+1}\). We consider bar framework consisting of the \(n\)-simplex and two vertices that are adjacent to one of the vertices of the \(n\)-simplex. This framework has exactly one
stress, which gives us one equation (from \((p')^T \Omega p = 0\)). From all such bar frameworks, we get \(\binom{n+1}{2}\) equations

\[\alpha |a_i|^2 + \alpha |a_j|^2 + \beta |a'_i - a'_j|^2 = 0\]

where \(\beta\) equals the stress between the two vertices and \(\alpha\) equals the sum of the stresses coming from the vertex \(q\) minus \(\beta\). We want to consider the framework in \(\mathbb{E}^{2n-1}\), so we let \(a'_i = (0, 0, \ldots, x_1, x_2, \ldots, x_{n-1})\). Thus, we can consider the \(a'_i\) to be points in \(\mathbb{E}^{n-1}\).

Manipulating the equation, we get that \(a'_i\) lies on an \((n-2)\)-sphere of radius \(\sqrt{k^2 - 1}|a'_j|\) with center \(ka'_j\). This means that the \((n-2)\)-sphere is orthogonal to the \((n-2)\)-sphere centered at the origin with radius \(|a'_j|\).

We again have 2 cases, depending on whether \(k\) is positive or negative.

**Case 1.** \(k \geq 0\)

Then \(a'_i\) lies inside the \((n-2)\)-sphere centered at \(ka'_j\).

We can assume that the \(a'_i\) are in general position — if not, then \(k\) of the \(a'_i\) lie on an affine subspace of dimension less than \(k-2\), and the argument for the spheres in dimension \(k-2\) gives a contradiction. Since the \(a'_i\) are in general position, the intersection of \(n-1\) of the spheres is two of the \(a'_i\).

We will get a contradiction if we consider the union of the \(n+1\) \((n-2)\)-spheres and their interiors. Each \(a'_i\) lies in the interior of the union, since each \(a'_i\) lies in the interior of one of the sphere. The boundary of this union contains none of the \(a'_i\), but this is a contradiction, since the \(a'_i\) lie on the intersection of the \((n-2)\)-spheres.

**Case 2.** \(k < 0\)

Here, we consider the intersection of the \(n+1\) \((n-2)\)-spheres. Since any \(n\) of the spheres have a nonempty intersection, the intersection of all \(n+1\) of the spheres is nonempty by Helly’s Theorem. Each \(a'_i\) is disjoint from the intersection, since it lies outside of one of the spheres. Thus, the boundary of the intersection contains none of the \(a'_i\), but this is a contradiction, since the \(a'_i\) lie on the intersection of the \((n-2)\)-spheres.

Thus, the \(n\)-simplex with flaps is second order rigid in \(\mathbb{E}^{2n-1}\). \(\square\)

Thus, we have proven Theorem 2, as the simplex with flaps provides a configuration in \(\mathbb{E}^n\), where a contraction exists, but there is not a continuous contraction in \(\mathbb{E}^{2n-1}\).
References


