Hilbert Sequences of Monomial Ideals

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Abstract

In this project, we investigate the Hilbert function of polynomial rings and various monomial ideals in these rings. We then use the results of these functions to form something called a Hilbert sequence. Our ultimate goal is to characterize all of the Hilbert sequences that arise using different polynomial rings and monomial ideals. In particular, we find the types of sequences that have a finite number of non-zero entries and a group of sequences that are guaranteed to be symmetric. Finally, we are able to describe all possible Hilbert sequences for every monomial ideal in the polynomial rings in either one or two variables. As it turns out, these are also all of the possible Hilbert sequences for all homogeneous ideals in the same polynomial rings.

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Acknowledgments

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1
An Introduction

When I began my senior project last fall, I started by investigating some basic properties of simplicial complexes. I knew that I was interested in a project involving algebra in some form, but since algebra is a very broad and comprehensive area, I had no idea exactly where to start. My advisor, Lauren Rose, helped me by giving me several ideas to work with and allowed me to select the one that I found to be the most intriguing. After computing several examples in each of these areas, I decided that the problems involving simplicial complexes seemed appealing. Ultimately, my studies in this topic brought me to learn about Hilbert functions of standard graded k-algebras. In particular, there is a way to construct a monomial ideal that corresponds to a given simplicial complex. When the Hilbert series of this ideal is computed, the numbers that arise turn out to be important invariants of the original simplicial complex.

It was around this time that I found myself to be more interested in studying the Hilbert functions of monomial ideals than I was in working on problems involving simplicial complexes. Specifically, the Hilbert function of the quotient of a polynomial ring by a monomial ideal can be viewed as a sequence of non-negative integers. I was very curious about the patterns that arose in these sequences and I decided that this was what I wanted to focus on for my senior project.

As it turns out, if you know the Hilbert functions of monomial ideals, you also know all Hilbert functions for any homogeneous ideal in the polynomial ring $k[x_1, \ldots, x_n]$. I state the following theorem and refer the reader to [3, Chapter 9, §3, Proposition 9] for a more thorough treatment of the subject.

Proposition 1.0.1. *Let $I \subset k[x_1, \ldots, x_n]$ be a homogeneous ideal and let $>$ be a graded monomial order on $k[x_1, \ldots, x_n]$. Then the monomial ideal generated by the leading terms of the polynomials in I has the same Hilbert function as I.*

Of course I have neither addressed what a homogeneous ideal is, nor have I explained what a graded monomial order is and how we can use one to define the leading term of a polynomial, (unfortunately, I will not be able to elaborate upon monomial orderings in the pages that follow but I refer the reader to [3] for more information). However, in terms of the material that is addressed in this project, the importance of the proposition above is that it tells us that every homogeneous ideal in $R = k[x_1, \dots, x_n]$ has a Hilbert sequence that is the same as the Hilbert sequence of some monomial ideal in R. Thus if we can find all possible Hilbert sequences of monomial ideals in R, we have really found all possible Hilbert sequences for all homogeneous ideals in R. This shows why understanding monomial ideals is particularly important.

In Chapter 4, I am able to prove the following theorem using Proposition 1.0.1 together with the results of my research. This theorem is the main result of my senior project.

Theorem. *Suppose $R = k[x, y]$. A sequence S is a Hilbert sequence for R/I where I is some homogeneous ideal in R if and only if*

$$S = (1, 2, 3, 4, 5, \dots, n, n_1, n_2, n_3, n_4, n_5, \dots)$$

for some $n \in \mathbb{N}$ with $n \geq n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq \dots$.

Once I had finally decided what I wanted to concentrate my efforts upon, I set out to find as much as I could about the different types of Hilbert functions that can arise in polynomial rings in one and more variables. This project was well suited to by mathematical ability because I was able to compute lots of examples early on without an extensive knowledge of the algebraic structure I was working within. My ultimate goal was to be able to classify all possible Hilbert sequences of monomial ideals in $k[x_1, \dots, x_n]$, and I was able to do this for $k[x]$ and $k[x, y]$. Along the way, I also found and proved results regarding Hilbert functions of monomial ideals in $k[x_1, \dots, x_n]$ for any $n \geq 1$.

In Chapter 2, the algebraic groundwork is laid out for the entire project. This chapter contains no original results, but it allows me to assume background material throughout the rest of the project. As I mentioned earlier, the examples I was computing early on in my research were not difficult to work through, but the actual definitions and results that I needed to use in order to give validity to my examples were strange and foreign to me at first. Thus Chapter 2 works through many of these definitions, beginning with some very basic definitions from linear algebra, and will hopefully give the reader a solid idea of how standard graded k-algebras work as well as how to work with them. This is a difficult chapter to work through, but one should keep in mind that a standard graded k-algebra is usually either $k[x_1, \dots, x_n]$ or $k[x_1, \dots, x_n]/I$ for some ideal I.

Chapter 3 introduces the Hilbert function, the Hilbert series, and the Hilbert sequence. It is a continuation of the second chapter in the sense that gives us a procedure to apply to the structure we established in Chapter 2. Since the crux of my project is seeded in these concepts, this chapter involves a lot of examples and should help to clarify in the reader's mind exactly how one goes about computing the Hilbert function and Hilbert sequence given a polynomial ring and a monomial ideal. If the reader retains exactly one piece of information from this chapter, I hope that it is the result of Theorem 3.1.6, a consequence of a simple fact from linear algebra.

All of my own research is presented in Chapter 4. I begin this chapter by addressing the importance of a monomial ideal and then prove some results regarding Hilbert functions of monomial ideals in $k[x_1, \ldots, x_n]$. These results are nice as they are guaranteed to work with any number of indeterminates. Then I abandon full generality to devote some attention to $k[x]$ and $k[x, y]$, two polynomial rings that are much easier to work in. Finally, I am able to give a complete characterization of all possible Hilbert sequences in $k[x]$ and $k[x, y]$. All of the work done in the previous chapters is put to use in this chapter and so Chapter 4 is a culmination of everything that comes before it.

Finally, Chapter 5 presents a list of open questions. These were questions that had that surfaced during my research of the past year, but due to the finite nature of the senior project, I was unable to devote a great deal of time to finding answers to them. They are questions that I find to be particularly interesting and if anyone can figure any of them out, please let me know. After all, if I was able to answer all of the questions that I encountered along the way, then I did not choose a very interesting project.

In closing, I have enjoyed working on this project and I hope that the reader finds it equally enjoyable to read. This project has given me a sound idea of what it is like to conduct research. More importantly, it has shown me exactly how to present this research to others in a coherent manner, a skill that will prove to be useful in the future. I find this to be the intent of undertaking a senior project, and in the process, I have learned more than I had ever expected.

2
The Algebraic Preliminaries

We begin with a brief review of some properties of vector spaces. We will then use these very broad definitions and theorems to help us define graded k-algebras and particular properties of these algebras. Next, we will define a graded ideal of a graded k-algebra and then examine the resulting quotient space. Finally, we compute some examples. We need this particular algebraic structure to define the Hilbert Functions, Hilbert Sequence and Hilbert Series of a standard graded k-algebras which we will elaborate upon in the following chapter.

2.1 Vector Spaces

Recall the following definition from linear algebra.

Definition. Let V be a vector space over a field k and let W_1 and W_2 be subspaces of V. Then the set $W_1 + W_2 = \{x + y \mid x \in W_1 \text{ and } y \in W_2\}$ is called the **sum** of W_1 and W_2. If $W_1 \cap W_2 = \{0\}$ we write $W_1 \oplus W_2$ and call this the **direct sum** of W_1 and W_2. △

Proposition 2.1.1. *Let W_1 and W_2 be subspaces of a vector space V over a field k. Suppose $V = W_1 \oplus W_2$. Then every $v \in V$ has a unique representation $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.*

Proof. We begin by showing that $0 \in V$ has a representation of the form $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. By definition, $0 \in W_1$ and $0 \in W_2$ so $0 = 0 + 0$. We will now show that $0 \in V$ has a unique representation. Let $0 = w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_2 = -w_1 \in W_1$. Therefore $w_2 \in W_1 \cap W_2$ so $w_2 = 0$ and thus $w_1 = 0$.
Now let $v \in V$. Then by definition, $v = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$.

Suppose that v can be represented as $w_1 + w_2$ and $u_1 + u_2$ for $w_1, u_1 \in W_1$ and $w_2, u_2 \in W_2$. Thus $w_1+w_2 = u_1+u_2$ so $w_1+w_2-u_1-u_2 = (w_1-u_1)+(w_2-u_2) = 0$. Since we have already shown that 0 has a unique representation in V, we conclude that $w_1 - u_1 = 0$ and $w_2 - u_2 = 0$ so then $w_1 = u_1$ and $w_2 = u_2$. □

Proposition 2.1.2. *Let W be a vector space over a field k with subspaces $W_0, W_1, \ldots, W_n$ for some $n \in \mathbb{N}$ and suppose that $W = W_0 \oplus W_1 \oplus \cdots \oplus W_n$. Then every element $w \in W$ has a unique representation $w = w_0 + w_1 + \cdots + w_n$ with $w_i \in W_i$ for $0 \le i \le n$.*

Proof. We will prove this by induction on n. If $n = 0$, the result is obvious. If $n = 1$, then $W = W_0 \oplus W_1$. By Proposition 2.1.1, every $w \in W$ has a unique representation $w = w_0 + w_1$ where $w_0 \in W_0$ and $w_1 \in W_1$, so the result holds. Now assume that the result holds for n. Let $w = w_0 + w_1 + \cdots + w_n + w_{n+1} = v_0 + v_1 + \cdots + v_n + v_{n+1}$ be two representations of $w \in W = W_0 \oplus W_1 \oplus \cdots \oplus W_n \oplus W_{n+1}$ with $w_i, v_i \in W_i$ for $0 \le i \le n+1$. If we let $B = W_0 \oplus W_1 \oplus \cdots \oplus W_n$, then $W = B \oplus W_{n+1}$ and so $w \in B \oplus W_{n+1}$. By Proposition 2.1.1, we have $w_0 + w_1 + \cdots + w_n = v_0 + v_1 + \cdots + v_n$ and $w_{n+1} = v_{n+1}$. From our inductive hypothesis, we conclude that $w_i = v_i$ for $0 \le i \le n$ and so w has a unique representation in W. □

Definition. Let V be a vector space over the field k and suppose that V has a set of vector subspaces $\{W_n\}_{n=0}^{\infty}$ indexed by the natural numbers. We define

$$W = \{w \in V \mid \text{there exists } r \in \mathbb{N} \text{ and } w_i \in W_i \text{ such that } w = w_0 + w_1 + \cdots + w_r\}$$

to be the **sum** of $W_0, W_1, W_2, \ldots$, and we write $W = W_0 + W_1 + W_2 + \cdots$. If $W_i \cap W_j = \{0\}$ for all $i, j \ge 0$ then we write $W = W_0 \oplus W_1 \oplus W_2 \oplus \cdots = \oplus_{n \ge 0} W_n$ and call W the **direct sum** of $W_0, W_1, W_2, \ldots$. △

Proposition 2.1.3. *Let W be a vector space over the field k with vector subspaces $\{W_n\}_{n=0}^{\infty}$ indexed by the natural numbers such that $W = \oplus_{n\ge 0} W_n$. Suppose that $w_0 + w_1 + \cdots + w_r = v_0 + v_1 + \cdots + v_s \in W$ with $r \le s$, and where $w_i \in W_i$ for $0 \le i \le r$ and $v_j \in W_j$ for $0 \le j \le s$. Then $w_i = v_i$ for $i \in \{0, \ldots r\}$ and $v_j = 0$ for $j \in \{r+1, \ldots, s\}$.*

Proof. Let $w = w_0 + w_1 + \cdots + w_r = v_0 + v_1 + \cdots + v_s$. Note that we can write $w = w_0 + w_1 + \cdots + w_r + w_{r+1} + \cdots + w_s$ with $w_{r+1} = \cdots = w_s = 0$ and so each $w_i \in W_i$ for $0 \le i \le s$. Now we have

$$\begin{aligned} w &= w_0 + w_1 + \cdots + w_r + 0 + \cdots + 0 \\ &= v_0 + v_1 + \cdots + v_r + v_{r+1} + \cdots + v_s \in W_0 \oplus W_1 \oplus \cdots \oplus W_s. \end{aligned}$$

By Proposition 2.1.2, we see that $w_0 = v_0$, $w_1 = v_1$, …, $w_r = v_r$ and $v_{r+1} = \cdots = v_s = 0$, as desired. □

Theorem 2.1.4. *Let V be a finite-dimensional vector space over a field k and suppose that W is subspace of V. Then W is also finite-dimensional and $dim(W) \leq dim(V)$. If $dim(W) = dim(V)$, then $V = W$.*

Proof. This is a standard result in linear algebra. For example, see [5, Theorem 1.11] □

The following definition is generally given in terms of groups, but it will be beneficial to us to present it in terms of vector spaces for the sake of Corollary 2.1.7.

Definition. Let W be a subspace of a vector space V over a field k and suppose $v \in V$. The **coset** of W **containing** v is defined to be the set

$$\overline{v} = v + W = \{v + w \mid w \in W\}.$$

The **quotient space of V modulo W** is defined to be the set

$$V/W = \{v + W \mid v \in V\}.$$

△

It is easy to show that V/W is also a vector space under the following operations:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \text{ for all } v_1, v_2 \in V,$$

and

$$a(v + W) = av + W \text{ for all } v \in V \text{ and } a \in k.$$

See, for example [5].

Proposition 2.1.5. *Let W be a subspace of a vector space V over a field k and let $v \in V$. Then $v + W = W$ if and only if $v \in W$.*

Proof. ($\Rightarrow$) Suppose that $v + W = W$. Since $0 \in W$ then $v = v + 0 \in W$.
($\Leftarrow$) Suppose that $v \in W$ and let $a \in v + W$. Then $a = v + w$ where $w \in W$ and so $a \in W$. Hence $v + W \subseteq W$. Now let $b \in W$. Since $v \in W$ then $c = b + v \in W$. Since W is a vector subspace, we have $c - 2v \in W$ and so $b = c - v = v + (c - 2v)$. Thus $b \in v + W$ and so $W \subseteq v + W$. We conclude that $v + W = W$. □

Theorem 2.1.6. *Let W be a subspace of a finite-dimensional vector space V over a field k. Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for W and let $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots v_m\}$ be an extension of this basis to a basis for V. Then $\{v_1 + W, v_2 + W, \ldots, v_m + W\}$ is a basis for V/W.*

Proof. Let $\beta = \{v_1 + W, v_2 + W, \ldots, v_m + W\}$. We will first show that β is linearly independent in V/W. Suppose $c_1, c_2, \ldots, c_m \in k$ and assume that

$$c_1(v_1 + W) + c_2(v_2 + W) + \cdots + c_m(v_m + W) = 0 + W,$$

the zero coset. Since $0 \in W$, by Proposition 2.1.5 we have $0 + W = W$ and so $c_1(v_1 + W) + c_2(v_2 + W) + \cdots + c_m(v_m + W) = W$. Note that $c_i(v_i + W) = c_i v_i + W$ for all $1 \leq i \leq m$. Hence

$$\begin{aligned}&(c_1v_1 + W) + (c_2v_2 + W) + \cdots + (c_mv_m + W) = \\ &= c_1(v_1 + W) + c_2(v_2 + W) + \cdots + c_m(v_m + W) = W.\end{aligned}$$

By definition, $\sum_{i=1}^{m} c_iv_i + W = (c_1v_1 + W) + (c_2v_2 + W) + \cdots + (c_mv_m + W) = W$ and so by Proposition 2.1.5, we have $\sum_{i=1}^{m} c_iv_i \in W$. However, the set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ is linearly independent and so $\sum_{i=1}^{m} c_iv_i = 0$ and $c_i = 0$ for $1 \leq i \leq m$. We conclude that β is linearly independent.

Now we must show that β spans V/W. Let $x \in V/W$. Then $x = v + W$ for some $v \in V$. Note that

$$v = \sum_{i=1}^{n} a_iu_i + \sum_{j=1}^{m} b_jv_j$$

for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in k$. Since $\sum_{i+1}^{n} a_iu_i \in W$ for all $1 \leq i \leq n$, we have

$$\begin{aligned}x = v + W &= (\sum_{i=1}^{n} a_iu_i + \sum_{j=1}^{m} b_jv_j) + W = \sum_{j=1}^{m} b_jv_j + W \\ &= \sum_{j+1}^{m} b_j(v_j + W) = b_1(v_1 + W) + b_2(v_2 + W) + \cdots + b_m(v_m + W).\end{aligned}$$

We conclude that β is a basis for V/W. □

Corollary 2.1.7. *Let V be a vector space over a field k and W a subspace of V. Then dim$(V/W) =$ dim$(V) -$ dim(W).*

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for W and let $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots v_m\}$ be an extension of this basis to a basis for V. By Theorem 2.1.6, we know that $\{v_1 + W, v_2 + W, \ldots, v_m + W\}$ is a basis for V/W. Hence $\dim(V) = n + m$, $\dim(W) = n$ and $\dim(V/W) = m$. Clearly $\dim(V/W) = m = n + m - n = \dim(V) - \dim(W)$. □

2.2 Standard Graded k-Algebras

We are now ready to define standard graded k-algebras.

Definition. A vector space A over a field k is called a **commutative k-algebra** if A possesses multiplication such that for every $x, y, z \in A$ and $\alpha, \beta \in k$ then

1. $xy = yx$;

2. $x(yz) = (xy)z$;

3. $x(y + z) = xy + xz$;

4. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

5. $\alpha(\beta x) = (\alpha\beta)x$.

For the purposes of this project, a **k-algebra** is a commutative k-algebra. Note that k is subring of A.

If a k-algebra A has a decomposition (as a vector space over k)

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

with

1. $A_0 = k$ and

2. $A_iA_j \subseteq A_{i+j}$ for all $i, j \geq 0$,

then A is called a **graded k-algebra** and we write $A = \oplus_{n\geq 0}A_n$. If there exists $r \in \mathbb{N}$ such that $A_n = 0$ for all $n > r$, we write $A = \oplus_{n=1}^{r} A_n$, and A is called **Artinian**. △

As it turns out, much of our work with graded k-algebras is motivated by rings of polynomials. We will be working with these rings throughout this project so it is important that we first make several definitions regarding some properties of polynomials. Note that for the remainder of this project, the set of natural numbers, denoted $\mathbb{N}$, includes 0.

Definition. Let $k[x_1, \dots, x_n]$ denote the ring of polynomials in the indeterminates $x_1, \dots, x_n$ over a field k. A **monomial** in $x_1, x_2, \dots, x_n$ is a product of the form $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}$. We will often write $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} = \mathbf{x}^\alpha$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$. The **total degree** of $\mathbf{x}^\alpha$ is $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ which we will denote by $|\alpha|$. A **polynomial** f in $x_1, x_2, \dots, x_n$ with coefficients in a field k is a finite k-linear combination of monomials. The **total degree of f** is the maximum total degree of all of the monomials that make up f. The polynomial f is said to be **homogeneous** if every monomial in f has the same total degree. If every monomial in f has the same total degree n, then we say that f is **homogeneous of degree** n. △

Example 2.2.1. Consider the ring $\mathbb{R}[x, y, z]$. Then by definition, $x^4y^5z^2 = \mathbf{x}^{(4,5,2)}$, $x^7y^3z = x^7y^3z^1 = \mathbf{x}^{(7,3,1)}$, $z = x^0y^0z^1 = \mathbf{x}^{(0,0,1)}$ and $1 = x^0y^0z^0 = \mathbf{x}^{(0,0,0)}$ are all monomials in x, y, z. The total degree of $x^4y^5z^2$ is $4 + 5 + 2 = 11$, the total degree of x^7y^3z is $7+3+1 = 11$, the total degree of z is $0+0+1 = 1$, and the total degree of 1 is $0 + 0 + 0 = 0$. Then $f = 3x^4y^5z^2 + \frac{2}{3}x^7y^3z + z + \sqrt{2}$, $g = \pi x^4y^5z^2 - 4x^7y^3z$, and $h = -5z + \frac{4}{5}$ are all polynomials in x, y, z with coefficients in $\mathbb{R}$. Thus $f, g, h \in$

$\mathbb{R}[x, y, z]$. The total degree of f is 11, the total degree of g is also 11 and the total degree of h is 1. However, g is the only polynomial of the three that is homogeneous since the two terms that make up g both have monomials with total degree 11. Thus g is homogeneous of degree 11. ◊

Example 2.2.2. Let $k[x]$ denote the ring of polynomials in the indeterminate x over a field k. Any polynomial in this ring can be written in the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for some $n \geq 0$ where $a_0, a_1, a_2, \ldots, a_n \in k$. Thus, it is easy to see that we can write

$$k[x] = R_0 + R_1 + R_2 + R_3 + \cdots$$

where $R_0 = \{a_0 \mid a_0 \in k\}$, $R_1 = \{a_1x \mid a_1 \in k\}$, $R_2 = \{a_2x^2 \mid a_2 \in k\}$, $R_3 = \{a_3x^3 \mid a_3 \in k\}$, and so forth with $R_n = \{a_nx^n \mid a_n \in k\}$. Now consider $R_i = \{a_ix^i \mid a_i \in k\}$ and $R_j = \{a_jx^j \mid a_j \in k\}$ for some $i \neq j$. Note that every element in R_i can be written as a_ix^i for some $a_i \in k$. Similarly, every element in R_j can be written as a_jx^j for some $a_j \in k$. Since $i \neq j$, the only time that we have $a_ix^i = a_jx^j$ is when $a_i = a_j = 0$ and so $R_i \cap R_j = \{0\}$. Hence

$$k[x] = R_0 \oplus R_1 \oplus R_2 \oplus R_3 \oplus \cdots .$$

Clearly $R_0 = k$ and if $a_ix^i \in R_i$ and $a_jx^j \in R_j$, then $(a_ix^i)(a_jx^j) \in R_iR_j$. Also, we have $(a_ix^i)(a_jx^j) = (a_ia_j)x^ix^j = (a_{i+j})x^{i+j}$ where $a_ia_j = a_{i+j} \in k$ and so $(a_ix^i)(a_jx^j) \in R_{i+j}$. Thus $R_iR_j \subseteq R_{i+j}$. We see that $k[x]$ is a graded k-algebra. ◊

Example 2.2.3. Let $R = k[x_1, \ldots, x_n]$ be the ring of polynomials in n indeterminates over the field k. We will prove in Theorem 2.2.5 that $R = \oplus_{n\geq 0}R_n$ is a graded k-algebra where R_i consists of all homogeneous polynomials of degree i for $i \geq 0$. ◊

Intuitively, we know that the coefficients of any polynomial uniquely determine that particular polynomial. Once we know that $k[x_1, \ldots, x_n]$ is a graded k-algebra, the reason for this will follow from the next proposition.

Proposition 2.2.4. *Let $A = \oplus_{n\geq 0}A_n$ be a graded k-algebra. Then for each element $a \in A$, there exists $r \in \mathbb{N}$ such that a can be represented as $a = a_0 + a_1 + a_2 + \cdots + a_r$ with $a_i \in A_i$ for $0 \leq i \leq r$, and the representation is unique for any such r.*

Proof. Since every graded k-algebra is a ring that contains the field k as a subring, then A is also a vector space over k. Hence this proposition follows directly from Proposition 2.1.3. □

Definition. Let $A = \oplus_{n\geq 0}A_n$ be a graded k-algebra. Each $a \in A_n$ is said to be **homogeneous of degree** n and we write $\deg(a) = n$. Note that $0 \in A_n$ for all $n \geq 0$ so $\deg(0)$ is arbitrary. We say that A is **finitely generated** if there exists a finite set of homogeneous elements $\{a_i\}_{1\leq i\leq m}$ such that A is spanned as a vector space over k by the monomials

$$\{a_1^{e_1}a_2^{e_2} \cdots a_m^{e_m} \mid e_i \in \mathbb{N} \text{ for } 1 \leq i \leq m\}.$$

If $\deg(a_i) = 1$ for all $i \geq 0$, then A is called a **standard graded k-algebra**. $\triangle$

It follows that $A_i = \text{span}\{a_1^{e_1} a_2^{e_2} \cdots a_m^{e_m} \mid e_1 + e_2 + \cdots + e_m = i\}$. Notice that the definition of homogeneous here is consistent with our definition of homogeneous polynomials.

Theorem 2.2.5. *Let k be a field and $R = k[x_1, \ldots, x_n]$ with deg(x_i) $= 1$ for all $1 \leq i \leq n$. Then R is a finitely generated standard graded k-algebra.*

Proof. Let $R_m = \text{span}\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N} \text{ and } \sum_{i=0}^{n} \alpha_i = m\} = \text{span}\{\mathbf{x}^\alpha \mid |\alpha| = m\}$. Note that since $\deg(x_i) = 1$ for all $1 \leq i \leq n$, then every element in R_m is a linear combination over k of monomials in $x_1, x_2, \ldots, x_n$ of total degree m. We begin by showing that $\oplus_{m \geq 0} R_m$ is a graded k-algebra. Let $i, j \in \mathbb{N}$ with $i \neq j$. Consider $R_i = \text{span}\{\mathbf{x}^\alpha \mid |\alpha| = i\}$ and $R_j = \text{span}\{\mathbf{x}^\beta \mid |\beta| = j\}$. Note that $0 \in R_i \cap R_j$ when all of the coefficients of any combination of monomials from either set are zero. However, 0 is the only element these two sets have in common and so $R_i \cap R_j = \{0\}$. Thus, the direct sum is well-defined. If $i = 0$, then $R_0 = \text{span}\{\mathbf{x}^\alpha \mid |\alpha| = 0\}$. In this case, since $\alpha_i \geq 0$ for $1 \leq i \leq n$, then $\mathbf{x}^\alpha = x_1^0 x_2^0 \cdots x_n^0 = 1$. Hence $R_0 = \text{span}\{1\} = k$. With full generality, suppose again that $i \in \mathbb{N}$. Let $a \in R_i$ and $b \in R_j$. Then a is a linear combination over k of monomials of total degree i and b is a linear combination over k of monomials of total degree j. Let $a' = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ be a monomial in a and $b' = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ be a monomial in b. Then a' has total degree $\alpha_1 + \alpha_2 + \cdots + \alpha_n = i$ and b' has total degree $\beta_1 + \beta_2 + \cdots + \beta_n = j$. Since $a'b' = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} = x_1^{\alpha_1+\beta_1} x_2^{\alpha_2+\beta_2} \cdots x_n^{\alpha_n+\beta_n}$ has total degree $(\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) + \cdots + (\alpha_n + \beta_n) = i + j$, it follows that ab is a linear combination over k of monomials of total degree $i + j$ and thus $R_i R_j \subseteq R_{i+j}$. We conclude that $\oplus_{m \geq 0} R_m$ is a standard graded k-algebra. Now we only need to show that $R = \oplus_{m \geq 0} R_m$. By Proposition 2.2.4, if $r \in \oplus_{m \geq 0} R_m$, then r can be written uniquely as $r_0 + r_1 + r_2 + \cdots + r_n$ for some $n \in \mathbb{N}$ where $r_i \in R_i$ for $0 \leq i \leq n$. However, each $r_i \in R_i$ is a linear combination over k of monomials in $x_1, x_2, \cdots, x_n$ with total degree i. It follows that if $r_n \neq 0$, then by definition, r is a polynomial of total degree n and so $r \in k[x_1, \ldots, x_n]$. Thus $\oplus_{m \geq 0} R_m \subseteq R$. Now let $f \in R$ be a polynomial of total degree s. We can rewrite f in terms of its homogeneous components so that $f = f_0 + f_1 + f_2 + \cdots + f_s$ where each f_i is the sum of all terms in f of total degree i. Then f_i is simply a linear combination over k of monomials in $x_1, x_2, \ldots, x_n$ of total degree i so thus $f_i \in R_i$ for $0 \leq i \leq s$. We conclude that $f \in \oplus_{m \geq 0} R_m$ and so $R \subseteq \oplus_{m \geq 0} R_m$. Therefore, $R = \oplus_{m \geq 0} R_m$. By hypothesis, $\deg(x_i) = 1$ for $1 \leq i \leq n$ and since R is spanned as a vector space over k by the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha_k \in \mathbb{N}$ for $1 \leq k \leq n$, we conclude that R is a finitely generated standard graded k-algebra. $\square$

We will now show that the dimension of each of these R_i's is quite easy to compute provided that R is standard.

Proposition 2.2.6. *Let k be a field and $R = k[x_1, \ldots, x_n]$ with $deg(x_m) = 1$ for all $1 \leq m \leq n$. Let $R_i = span\{x_1^{a_1} \cdots x_n^{a_n} \mid a_j \in \mathbb{N}$ for $1 \leq j \leq n$, and $\sum_{j=1}^{n} a_j = i\}$. Then the dimension of R_i as a vector space over k is*

$$dim_k(R_i) = \binom{n+i-1}{i}.$$

Proof. Note that $\dim_k(R_i)$ is simply the total number of monomials of total degree i. These monomials take the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = i$. Since some of these α_j's may equal zero for $1 \leq j \leq n$, the number of monomials of this type is the number of distributions of i identical objects to n distinct recipients, which is $\binom{n+i-1}{i}$. For a combinatorial proof of this fact, see [2, Theorem 2.3]. □

Example 2.2.7. Let $R = k[x, y]$ with deg $(x) =$ deg $(y) = 1$. Then

$$R_0 = \text{span}\{1\},$$

$$R_1 = \text{span}\{x, y\},$$

$$R_2 = \text{span}\{x^2, xy, y^2\},$$

$$R_3 = \text{span}\{x^3, x^2y, xy^2, y^3\},$$

and so forth with

$$R_i = \text{span}\{x^i, x^{i-1}y, x^{i-2}y^2, \ldots, x^2y^{i-2}, xy^{i-1}, y^i\}.$$

Thus $\dim_k(R_0) = 1$, $\dim_k(R_1) = 2$, $\dim_k(R_2) = 3$ and $\dim_k(R_i) = i + 1$.

Proposition 2.2.6 tells us that $\dim(R_i) = \binom{2+i-1}{i} = \binom{i+1}{i} = i + 1$. This means that there are $i + 1$ monomials of total degree i. Indeed these monomials are

$$\{x^i, x^{i-1}y, x^{i-2}y^2, x^{i-3}y^3, \ldots, x^3y^{i-3}, x^2y^{i-2}, xy^{i-1}, y^i\}.$$

◊

Though it is sometimes interesting to consider $k[x_1, \ldots, x_n]$ where some of the x_i's have degree other than 1, these types of rings are extremely difficult to work with and so for the remainder of this project, we will assume that if $R = k[x_1, \ldots, x_n]$, then R is standard.

2.3 Graded Ideals

Since graded k-algebras are also rings, investigating the ideals and quotient rings of these algebras can sometimes provide us with interesting information. We begin by recalling the following definition.

Definition. A subring I of a ring R is called an **ideal** if $aI \subseteq I$ and $Ib \subseteq I$ for all $a, b \in R$. △

Also recall that if $a+I$ is the coset of an ideal I containing a, and $b+I$ is the coset of I containing b, then $(a+I)+(b+I)=(a+b)+I$ and $(a+I)(b+I)=(ab)+I$.

Proposition 2.3.1. *Let R be a ring with unity and let I be an ideal of R. Let $u \in R$ be a unit. If $u \in I$ then $I = R$ In particular, if 1_R denotes the multiplicative identity of R then $1_R \in I$ implies $I = R$.*

Proof. Let u be a unit of a ring R. Then there exists $u^{-1} \in R$ such that $uu^{-1} = u^{-1}u = 1_R$. Suppose $u \in I$. Then $1_R = uu^{-1} = u^{-1}u \in I$. If $1_R \in I$, then $a = 1_R \cdot a = a \cdot 1_R \in I$ for all $a \in R$. Thus $I = R$. □

It would seem natural to consider ideals I in a graded k-algebra that have a vector space decomposition $I = I_0 \oplus I_1 \oplus I_2 \oplus \cdots = \oplus_{n\geq 0} I_n$. It turns out that these ideals are very important.

Definition. If $A = \oplus_{n\geq 0} A_n$ is a graded k-algebra, a **graded ideal** I is a vector subspace of the form $I = \oplus_{n\geq 0} I_n$ such that

1. For all $n \geq 0$, $I_n \subseteq A_n$.
2. $A_i I_j \subseteq I_{i+j}$ for all $i, j \geq 0$.

△

Proposition 2.3.2. *Let $A = \oplus_{n\geq 0} A_n$ be a graded k-algebra. Let I be an ideal of A. Then I is a graded ideal if and only if given $a_0 + a_1 + \cdots + a_r \in I$ for some $r \in \mathbb{N}$ with $a_i \in A_i$, then $a_i \in I$ for $0 \leq i \leq r$.*

Proof. ($\Rightarrow$) Suppose $I = \oplus_{n\geq 0} I_n$ is a graded ideal of A. For some $r \in \mathbb{N}$, let $a = a_0 + a_1 + \cdots + a_r \in I$ where $a_i \in A_i$ for $0 \leq i \leq r$. Since I is a graded ideal, then we also have $a = b_0 + b_1 + \cdots + b_r$ where each $b_i \in I_i \subseteq A_i$ for all $0 \leq i \leq r$. By Proposition 2.2.4, a has a unique representation because $a \in I \subseteq A$. Hence $a_i = b_i \in I_i \subseteq I$ for all $0 \leq i \leq r$.

($\Leftarrow$) Suppose that for all $r \in \mathbb{N}$, whenever $a_0 + a_1 + \cdots + a_r \in I$ with each $a_i \in A_i$ for $0 \leq i \leq r$, then $a_i \in I$ as well. Define $I_j = A_j \cap I$ for all $j \geq 0$. We begin by showing that $I = I_0 + I_1 + I_2 + \cdots$. Since each $I_j \subseteq I$ for all $j \geq 0$, it follows that $I_0 + I_1 + I_2 + \cdots \subseteq I$. Now let $a \in I$. Since $I \subseteq A$, then $a \in A$. By Proposition 2.2.4, we know that there exists $m \in \mathbb{N}$ such that $a = a_0 + a_1 + \cdots + a_m$ and each $a_i \in A_i$ for $0 \leq i \leq m$. By hypothesis, each $a_i \in I$ as well. Therefore, each $a_i \in A_i \cap I = I_i$ and so $a \in I_0 + I_1 + I_2 + \cdots$. Thus $I \subseteq I_0 + I_1 + I_2 + \cdots$ and so we conclude that $I = I_0 + I_1 + I_2 + \cdots$.

We will now show that $I_i \cap I_j = \{0\}$ for all $i, j \geq 0$ with $i \neq j$. Note that $I_i \cap I_j = (A_i \cap I) \cap (A_j \cap I) = (A_i \cap A_j) \cap I = \{0\} \cap I = \{0\}$ whenever $i \neq j$. Therefore, $I = \oplus_{n\geq 0} I_n$.

Finally, we will show that I satisfies the properties of a graded ideal. Clearly $I_i \subseteq A_i$ for all $i \geq 0$ because $I_i = A_i \cap I \subseteq A_i$. Let $a_i b_j \in A_i I_j$ where $a_i \in A_i$ and

$b_j \in I_j$. Since $b_j \in A_j \cap I$, then $b_j \in A_j$ and $b_j \in I$. Hence $a_i b_j \in A_{i+j}$. Because I is an ideal, we also know that $a_i b_j \in I$ so then $a_i b_j \in A_{i+j} \cap I = I_{i+j}$. We conclude that $A_i I_j \subseteq I_{i+j}$ and thus I is a graded ideal. □

In the proof of the last proposition, we defined I_j to be the intersection of A_j with I when $A = \oplus_{n\geq 0} A_n$ is a graded k-algebra with an ideal I satisfying the properties given above. Then we showed that $I = \oplus_{n\geq 0} I_n$. As it turns out, for every graded ideal $I = \oplus_{n\geq 0} I_n$, $I_j = A_j \cap I$.

Proposition 2.3.3. *Let $A = \oplus_{n\geq 0} A_n$ be a graded k-algebra with a graded ideal $I = \oplus_{n\geq 0} I_n$. Then $I_i = A_i \cap I$ for all $i \geq 0$.*

Proof. By definition, $I_i \subseteq A_i$ and $I_i \subseteq I$. It follows that $I_i \subseteq A_i \cap I$ for all $i \geq 0$. Now let $a \in A_i \cap I$. Then $a \in I$ and since $I \subseteq R$, we have $a \in R$. By Proposition 2.2.4, there exists $r \in \mathbb{N}$ such that a can be written uniquely as $a = a_0 + a_1 + \cdots + a_r$ with $a_i \in A_i$ for $0 \leq i \leq r$. By Proposition 2.3.2, each $a_i \in I$ as well. However, we also know that $a \in A_i$ and so a can be uniquely represented as $a = \underbrace{0 + \ldots + 0}_{i-1 \text{ times}} + a'_i + \underbrace{0 + \ldots + 0}_{r-i \text{ times}}$ where $a'_i \in A_i$. Since $I_j \subseteq A_j$ for all $j \in \mathbb{N}$, we conclude that $a_0 = a_1 = \cdots = a_{i-1} = a_{i+2} = \cdots = a_r = 0$ and $a = a_i = a'_i$. Since $a = a_i \in I_i$, then $A_i \cap I \subseteq I_i$. Therefore, $I_i = A_i \cap I$. □

Proposition 2.3.4. *Let $A = \oplus_{n\geq 0} A_n$ be a graded k-algebra with a graded ideal $I = \oplus_{n\geq 0} I_n$. Then either $I_0 = \{0\}$ or $I = A$.*

Proof. By definition, $I_0 \subseteq A_0 = k$. Let $x \in I_0 \subseteq I$ with $x \neq 0$. Since k is a field then every non-zero element of k is a unit. By Proposition 2.3.1 we have $I = A$. In order to have the case where I is a proper ideal, no such x can be contained in I_0 and so $I_0 = \{0\}$. □

Definition. Let $A = \oplus_{n\geq 0} A_n$ be a graded k-algebra and let $a_1, a_2, \ldots a_m$ be elements of A. Then $I = \langle a_1, a_2, \ldots, a_m \rangle = \{\sum_{i=1}^m a_i b_i \mid b_i \in A \text{ for } 1 \leq i \leq m\}$ is called the **ideal generated by** $a_1, a_2, \ldots a_m$. △

It is an easy exercise to show that this set is an ideal. However, it is not so apparent that this ideal is also a graded ideal.

Proposition 2.3.5. *Let $A = \oplus_{n\geq 0}$ be a graded k-algebra. Suppose $a_1, a_2, \ldots, a_m \in A$ are homogeneous elements with deg$(a_i) = d_i$ for $1 \leq i \leq m$. Then $I = \langle a_1, a_2, \ldots, a_m \rangle$ is a graded ideal.*

Proof. Let $x \in I$. Then $x = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m$ where $b_1, b_2, \ldots, b_m \in A$. We can write each b_i as a sum of homogeneous components and so $b_i = b_{i_0} + b_{i_1} + \cdots + b_{i_k} =$

$\sum_{j=0}^{k} b_{i_j}$ for some $k \in \mathbb{N}$ where each b_{i_j} has total degree j. Note that $b_{i_j} \in A_j$. Thus

$$\begin{aligned} x &= a_1 \sum_{j=0}^{k_1} b_{1_j} + a_2 \sum_{j=0}^{k_2} b_{2_j} + \cdots + a_m \sum_{j=0}^{k_m} b_{m_j} \\ &= \sum_{j=0}^{k_1} a_1 b_{1_j} + \sum_{j=0}^{k_2} a_2 b_{2_j} + \cdots + \sum_{j=0}^{k_m} a_m b_{m_j}. \end{aligned}$$

Then each $a_i b_{i_j}$ is homogeneous of total degree $d_i + j$, and since $a_i \in I$, each $a_i b_{i_j} \in I$ as well. Let x_r be the sum of all $a_p b_{p_s}$ such that $d_p + s = r$. Then we can write $x = x_0 + x_2 + \cdots + x_t$ where t is the maximum total degree of all of the $a_i b_{i_j}$'s. Notice that each $x_q \in A_q$ and is homogeneous of degree q for $0 \leq q \leq t$. However, x_q is simply a sum of things in I and so $x_q \in I$. By Proposition 2.3.2, we conclude that I is a graded ideal. □

It turns out that if A is a graded k-algebra and I is a graded ideal of A, then their ring quotient A/I is also a graded k-algebra.

Theorem 2.3.6. *Let $A = \oplus_{n \geq 0} A_n$ be a k-graded algebra with a proper graded ideal $I = \oplus_{n \geq 0} I_n$. Then A/I is also a graded k-algebra with $A/I \cong \oplus_{n \geq 0}(A_n/I_n)$ as a vector space over k.*

Proof. We begin by showing that $\oplus_{n \geq 0}(A_n/I_n)$ is a graded k-algebra with multiplication defined as follows: if $\overline{a_i} \in A_i/I_i$ and $\overline{a_j} \in A_j/I_j$, then $\overline{a_i} \cdot \overline{a_j} = \overline{a_i a_j} \in A_{i+j}/I_{i+j}$. We must first make sure that this multiplication is well-defined. To do so, we shall show that if $a_i + I_i = b_i + I_i$ and $a_j + I_j = b_j + I_j$, then $a_i a_j + I_{i+j} = b_i b_j + I_{i+j}$. Notice that we want $(a_i a_j - b_i b_j) + I_{i+j} = 0 + I_{i+j}$ and so we only need to show that $a_i a_j - b_i b_j \in I_{i+j}$ to prove that this multiplication is well-defined. Note that we can write $a_i a_j - b_i b_j$ as

$$a_i a_j - b_i b_j + a_i b_j - a_i b_j = a_i(a_j - b_j) + b_j(a_i - b_i).$$

Since $a_i + I_i = b_i + I_i$, then $(a_i - b_i) + I_i = 0 + I_i$ and so $a_i - b_i \in I_i$. We can similarly show that $a_j - b_j \in I_j$. Since $a_i \in A_i$, by the properties of graded ideals we have $a_i(a_j - b_j) \in A_i I_j \subseteq I_{i+j}$. Similarly, since $b_j \in A_j$, then $b_j(a_i - b_i) \in A_j I_i \subseteq I_{i+j}$. Therefore, $a_i(a_j - b_j) + b_j(a_i - b_i) \in I_{i+j}$ and so $a_i a_j - b_i b_j \in I_{i+j}$, as was desired. We conclude that our multiplication is well-defined.

Since $A_0 = k$ and $I \neq A$, then by Proposition 2.3.4, $I_0 = \{0\}$. Hence $A_0/I_0 = k/\{0\} = k$. Using the multiplication we just defined, let $z = xy$ where $x \in A_i/I_i$ and $y \in A_j/I_j$. By definition, $z \in A_{i+j}/I_{i+j}$. Thus $\oplus_{n \geq 0}(A_n/I_n)$ is a graded k-algebra.

To show $A/I \cong \oplus_{n \geq 0}(A_n/I_n)$, we need to find a ring homomorphism $\phi\colon A \longrightarrow \oplus_{n \geq 0}(A_n/I_n)$ that is surjective with $\ker(\phi) = I$. If we can find such a map ϕ, then the First Isomorphism Theorem for rings says that there exists a map $\overline{\phi}\colon A/I \longrightarrow \oplus_{n \geq 0}(A_n/I_n)$ that is a ring isomorphism (see, for example [4] Theorem 4.1.2).

Define $\phi\colon A \longrightarrow \oplus_{n\geq 0}(A_n/I_n)$ as follows: if $a \in A$ with $a = a_0 + a_1 + \cdots + a_r$ for some $a_i \in A_i$ and $r \in \mathbb{N}$, then

$$\phi(a) = \overline{a_0} + \overline{a_1} + \cdots + \overline{a_r}$$

where $\overline{a_i} = a_i + I_i$.

We will first show that ϕ is a ring homomorphism. Let $a, b \in A$ and choose $p \in \mathbb{N}$ such that both a and b have the form $a = a_0 + a_1 + \cdots + a_p$ and $b = b_0 + b_1 + \cdots + b_p$ with $a_i, b_i \in A_i$ for $0 \leq i \leq p$. Then

$$\begin{aligned}
\phi(a+b) &= \phi\big((a_0 + a_1 + \cdots + a_p) + (b_0 + b_1 + \cdots + b_p)\big) \\
&= \phi\big((a_0 + b_0) + (a_1 + b_1) + \cdots + (a_p + b_p)\big) \\
&= \overline{(a_0 + b_0)} + \overline{(a_1 + b_1)} + \cdots + \overline{(a_p + b_p)} \\
&= (\overline{a_0}) + (\overline{b_0}) + (\overline{a_1}) + (\overline{b_1}) + \cdots + (\overline{a_p}) + (\overline{b_p}) \\
&= (\overline{a_0} + \overline{a_1} + \cdots + \overline{a_p}) + (\overline{b_0} + \overline{b_1} + \cdots + \overline{b_p}) = \phi(a) + \phi(b)
\end{aligned}$$

because of the definition of addition in cosets. Also, note that

$$ab = (a_0 + a_1 + \cdots + a_p)(b_0 + b_1 + \cdots + b_p) = \sum_{i,j=0}^{p} a_i b_j.$$

If $c_k = \sum_{i+j=k} a_i b_j$ for all $i, j \leq n$, then $ab = c_0 + c_1 + \cdots + c_{2p}$ where $c_i \in A_i$ for $0 \leq i \leq 2p$. Thus

$$\phi(ab) = \phi(c_0 + c_1 + \cdots + c_{2n}) = \overline{c_0} + \overline{c_1} + \cdots + \overline{c_{2n}}.$$

By the properties of cosets,

$$\begin{aligned}
\overline{c_k} &= \overline{a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0} \\
&= \overline{a_0 b_k} + \overline{a_1 b_{k-1}} + \cdots + \overline{a_{k-1} b_1} + \overline{a_k b_0} \\
&= (\overline{a_0})(\overline{b_k}) + (\overline{a_1})(\overline{b_{k-1}}) + \cdots + (\overline{a_{k-1}})(\overline{b_1}) + (\overline{a_k})(\overline{b_0}).
\end{aligned}$$

Notice that the last equality uses our definition of multiplication in $\oplus_{n\geq 0} A_n/I_n$. Therefore, $\phi(ab) = \sum_{i,j=0}^{p} (\overline{a_i})(\overline{b_j}) = \sum_{i=0}^{p} \overline{a_i} \sum_{j=0}^{p} \overline{b_j} = \phi(a)\phi(b)$. We conclude that ϕ is a ring homomorphism.

Let $x \in \oplus_{n\geq 0}(A_n/I_n)$. Since $\oplus_{n\geq 0}(A_n/I_n)$ is a graded k-algebra, then we can write $x = x_1 + x_2 + \cdots + x_r$ where $x_i \in A_i/I_i$. Hence $x_i = a_i + I_i$ for some $a_i \in A_i$, and so $x_i = \overline{a_i}$. Thus $\phi(a_1 + a_2 + \cdots + a_r) = x$ so ϕ is surjective. Now suppose $a = a_0 + a_1 + \cdots + a_r \in \ker(\phi)$. Then $\phi(a) = \phi(a_0 + a_1 + \cdots + a_r) = \overline{0}$ so $\overline{a_0} + \overline{a_1} + \cdots + \overline{a_r} = \overline{0}$. By Proposition 2.2.4, $\overline{a_0} = \overline{a_1} = \cdots = \overline{a_r} = \overline{0}$. Proposition 2.1.5 tells us that if $a_i + I_i = \overline{a_i} = \overline{0}$, then $a_i \in I_i$ for all $i \geq 0$. Thus $a \in I$ so $\ker(\phi) \subseteq I$. Now let $a \in I$ so $a = a_0 + a_1 + \cdots + a_r$ for some $a_i \in I_i$. Then $a_i + I_i = I_i$ and we deduce that $\overline{a_i} = \overline{0}$. Hence $\phi(a) = \phi(a_0 + a_1 + \cdots + a_r) = \overline{a_0} + \overline{a_1} + \cdots + \overline{a_r} = \overline{0} + \overline{0} + \cdots + \overline{0} = \overline{0}$. Therefore $a \in \ker(\phi)$. Since the choice of $a \in I$ was arbitrary, we conclude that $I \subseteq \ker(\phi)$. □

Throughout this project, we are mostly concerned with ideals of the form $\langle a_1, a_2, \ldots, a_m\rangle$ where $a_1, a_2, \ldots, a_m$ are monomials. We will now introduce some techniques for analyzing polynomial rings, monomial ideals and their corresponding quotient spaces. Some examples should help to clarify this.

Example 2.3.7. Consider $R = k[x, y]$, (remember that $\deg(x) = \deg(y) = 1$). Let $I = \langle x^3, x^2y^3, y^6\rangle = \{h_1x^3 + h_2x^2y^3 + h_3y^6 \mid h_1, h_2, h_3 \in R\}$. To figure out what I looks like, we first use some intuition. For example, if we let $h_2 = h_3 = 0$, then I contains h_1x^3 where h_1 is any polynomial (in particular, any monomial) in R. Similarly, any polynomial (or monomial) multiple of x^2y^3 or y^6 is also in I. We can write $I = \oplus_{n\geq 0} I_n$ where I_i is the span of all monomials of total degree i in I. Thus

$$I_0 = I_1 = I_2 = (0), \text{ (since no monomial of our ideal has total degree 0, 1, or 2)},$$

$$I_3 = \operatorname{span}\{x^3\}, \text{ (since } x^3 \in I),$$
$$I_4 = \operatorname{span}\{x^4, x^3y\} \text{ (since } x^4 = x(x^3) \text{ and } x^3y = y(x^3)),$$
$$I_5 = \operatorname{span}\{x^5, x^4y, x^3y^2, x^2y^3\}, \text{ (since } x^5 = x^2(x^3), x^4y = xy(x^3), x^3y^2 = y^2(x^3), \text{ and } x^2y^3 \in I).$$

Similarly, we can see that

$$I_6 = \operatorname{span}\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, y^6\},$$

$$I_7 = \operatorname{span}\{x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7\},$$
$$I_8 = \operatorname{span}\{x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8\},$$

and so on. Notice that $I_i = R_i$ (as defined in Example 2.2.7) for all $i \geq 7$.

Intuitively, we can construct R/I by deleting the monomial terms in each I_i from each R_i. The next example will illustrate this in a more convincing manner, but sometimes it is easier to visualize R/I in the manner that follows. Note that we write $\overline{x^ay^b}$ to indicate $x^ay^b + I_i$.

$$R_0/I_0 \cong k,$$

$$R_1/I_1 \cong \operatorname{span}\{\overline{x}, \overline{y}\},$$
$$R_2/I_2 \cong \operatorname{span}\{\overline{x^2}, \overline{xy}, \overline{y^2}\},$$
$$R_3/I_3 \cong \operatorname{span}\{\overline{x^2y}, \overline{xy^2}, \overline{y^3}\},$$
$$R_4/I_4 \cong \operatorname{span}\{\overline{x^2y^2}, \overline{xy^3}, \overline{y^4}\},$$
$$R_5/I_5 \cong \operatorname{span}\{\overline{xy^4}, \overline{y^5}\},$$
$$R_6/I_6 \cong \operatorname{span}\{\overline{xy^5}\},$$
$$R_7/I_7 \cong R_8/I_8 \cong R_9/I_9 \cong \cdots \cong (\overline{0}).$$

Note that since $R/I \cong \oplus_{n=1}^{6}(R_n/I_n)$, then R/I happens to be Artinian. ◊

In Chapter 4 we will show that for each coset $m + I$, m is the only monomial in that coset so we can misuse notation and write m instead of $\overline{m} = m + I$ in R/I.

Example 2.3.8. Let $R = k[x, y]$ and let $I = \langle x^2, xy\rangle$. Since R is a standard graded k-algebra, we can write $R = \oplus_{n\geq 0} R_n$ where

$$R_i = \text{span}\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_j \in \mathbb{N} \text{ for } 1 \leq i \leq n \text{ and } \sum_{j=1}^{n} \alpha_j = i\}.$$

Then $R_0 = \text{span}\{x^0y^0\} = \text{span}\{1\} = k$, $R_1 = \text{span}\{x, y\}$, $R_2 = \text{span}\{x^2, xy, y^2\}$, $R_3 = \text{span}\{x^3, x^2y, xy^2, y^3\}$, and so forth as outlined in the proof of Theorem 2.2.5 and Example 2.2.7.

Also note that we can write $I = \oplus_{n\geq 0} I_n$ where I_i is the span of all monomials in I of total degree i. Then

$$I_0 = I_1 = (0),$$
$$I_2 = \text{span}\{x^2, xy\},$$
$$I_3 = \text{span}\{x^3, x^2y, xy^3\},$$

and so on. In keeping with Theorem 2.3.6, we see that

$$R_0 = k \text{ and } I_0 = (0) \text{ implies that } R_0/I_0 \cong k,$$

$$R_1 = \text{span}\{x, y\} \text{ and } I_1 = (0) \text{ implies that } R_1/I_1 \cong R_1,$$

$$R_2 = \text{span}\{x^2, xy, y^2\} \text{ and } I_2 = \text{span}\{x^2, xy\} \text{ and } R_2/I_2 \cong \text{span}\{\overline{y^2}\},$$

$$R_3 = \text{span}\{x^3, x^2y, xy^2, y^3\} \text{ and } I_3 = \text{span}\{x^3, x^2y, xy^2\} \text{ implies that } R_2/I_2 \cong \text{span}\{\overline{y^3}\},$$

and so forth. We notice that $R_n/I_n \cong \text{span}\{\overline{y^n}\}$ for $n \geq 2$. ♢

Though we have only been computing examples in $k[x, y]$, we can compute examples in $k[x, y, z]$ or even $k[x_1, \ldots, x_n]$ in the exact same manner. However, the number of monomials tends to get large very quickly. For example, if $R = k[x_1, x_2, \ldots, x_{10}]$ then there are $\binom{10+2-1}{2} = \binom{11}{2} = 55$ monomials of total degree 2 in R_2!

2.4 A Visual Example in $k[x, y]$

We now shift our focus to the polynomial ring $R = k[x, y]$. In Chapter 4, we will formally define a monomial ideal, but for now, let us assume that a monomial ideal I in $k[x, y]$ is any ideal generated by a set of monomials in $k[x, y]$. Since every monomial $x^iy^j \in k[x, y]$ is homogeneous of degree $i + j$, we see that every monomial ideal is generated by a set of homogeneous elements. By Proposition 2.3.5, then every monomial ideal is a graded ideal. In Theorem 2.2.5 and previous examples, we have seen that

$$R_m = \text{span}\{x^i y^j \mid i + j = m\}$$

and

$$I_m = \text{span}\{x^i y^j \mid i + j = m \text{ and } x^i y^j \in I\}.$$

Suppose $x^a y^b \in I$. It follows that $x^a y^b \in I_{a+b}$ since $x^a y^b$ is homogeneous of degree $a + b$. Now consider I_{a+b+r} for some $r \in \mathbb{N}$. Notice that $(x^a y^b)(x^c y^d) \in I_{a+b+r}$ for all monomials $x^c y^d$ such that the total degree of $x^c y^d$ is r. By Proposition 2.2.6, there are $r + 1$ such monomials. For example, if $r = 1$, then there are two monomials of total degree 1 and these monomials are x and y. Thus, the monomials $(x^a y^b)(x) = x^{a+1} y^b$ and $(x^a y^b)(y) = x^a y^{b+1}$ are also in I. Similarly, if $r = 2$, then there are three monomials of total degree 2 and these monomials are $x^2, xy,$ and y^2. Therefore, the monomials $(x^a y^b)(x^2) = x^{a+2} y^b$, $(x^a y^b)(xy) = x^{a+1} y^{b+1}$ and $(x^a y^b)(y^2) = x^a y^{b+2}$ are all in I.

Therefore, we can make a graph that represents any monomial ideal I in $\mathbb{N} \times \mathbb{N}$. For example, we plot the point (p, q) if and only if $x^p y^q \in I$. More specifically, if $I = \langle x^r y^s \rangle$, then we simply plot the point (r, s) and every point $(r + t, s + u)$ such that $t, u \in \mathbb{N}$. Thus we plot (r, s) and every point to the right and above (r, s).

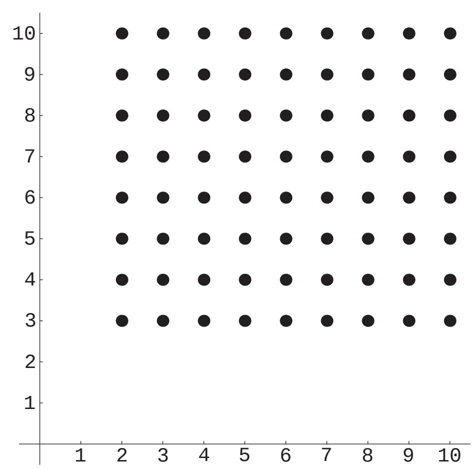

Figure 2.4.1.

Example 2.4.1. The graph associated with the monomial ideal $I = \langle x^2 y^3 \rangle$ is shown in Figure 2.4.1. ◊

As it turns out, if $I = \langle x^{a_1} y^{b_1}, x^{a_2} y^{b_2}, \ldots, x^{a_m} y^{b_m} \rangle$, then our graph consists of every point $(a_i + c, b_i + d) = (a_i, b_i) + (c, d)$ where $c, d \in \mathbb{N}$ for $1 \leq i \leq m$. This is the same as taking the union of the points in the graphs for $I_1 = \langle x^{a_1} y^{b_1} \rangle$, $I_2 = \langle x^{a_2} y^{b_2} \rangle$, up to $I_m = \langle x^{a_m} y^{b_m} \rangle$.

Example 2.4.2. The graph associated with the monomial ideal $I = \langle x^3, x^2y^3, y^6\rangle$ is shown in Figure 2.4.2. Notice that in this graph, we can see which monomials

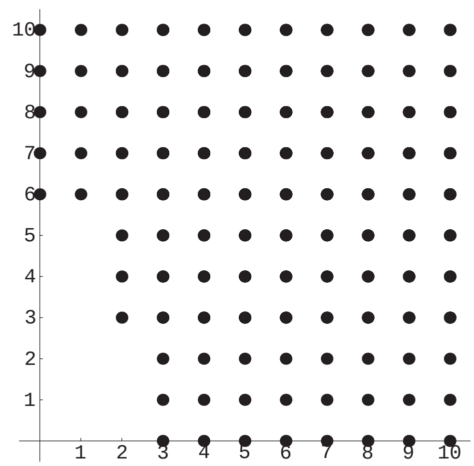

Figure 2.4.2.

generate I. They are represented by the points with no other points to the left and below them. These points are $(0, 6)$ $(2, 3)$, and $(3, 0)$ which correspond to the monomials $x^0y^6 = y^6$, x^2y^3 and $x^2y^0 = x^2$, respectively, and are the exact monomials we chose to generate I. $\Diamond$

These graphs can also show us which generators for I are redundant. For example, if $I_1 = \langle x^3, x^2y^3, y^6, x^8y\rangle$, the monomial x^8y is a redundant generator since $(8, 1) = (3, 0) + (5, 1)$. What this literally means is that $x^8y^1 = (x^3)(x^5y^1)$. Similarly, if $I_2 = \langle x^3, x^2y^3, y^6, x^9y^7\rangle$, then the monomial x^9y^7 is redundant since $(9, 7) = (2, 0) + (7, 7)$. In this case, we also find that $(9, 7) = (2, 3) + (7, 4)$ and $(9, 7) = (0, 6) + (9, 1)$ which means that $(9, 7)$ is a point contained in each of the graphs for the ideals $\langle x^2\rangle$, $\langle x^3y^2\rangle$, and $\langle y^6\rangle$.

In Chapter 4, we will show that if $x^ay^b \notin I$, then $\overline{x^ay^b} \in (R/I)$ and then we will show that it is acceptable to write $x^ay^b \in (R/I)_{a+b}$ for $\overline{x^ay^b}$. Thus we can also make a graph to represent R/I by plotting all points $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $x^my^n \notin I$. Notice that this is consistent with Corollary 2.1.7. If $I = \langle x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \ldots, x^{a_m}y^{b_m}\rangle$ then we can also make a graph for R/I by taking the intersection of all of the points in the graphs of $R/\langle x^{a_1}y^{b_1}\rangle$, $R/\langle x^{a_2}y^{b_2}\rangle$, up through $R/\langle x^{a_m}y^{b_m}\rangle$.

Example 2.4.3. Let $I = \langle x^3, x^2y^3, y^6\rangle$ as in Example 2.4.2. Then the graph associated with R/I is shown in Figure 2.4.3. Notice that there are a finite number of points in this graph. However, it need not always be the case that the graph associated with R/I has a finite number of points. $\Diamond$

Example 2.4.4. Let $I = \langle x^2, xy\rangle$. Then the graph associated with I is shown in Figure 2.4.4 and the graph associated with R/I is shown in Figure 2.4.5 on the

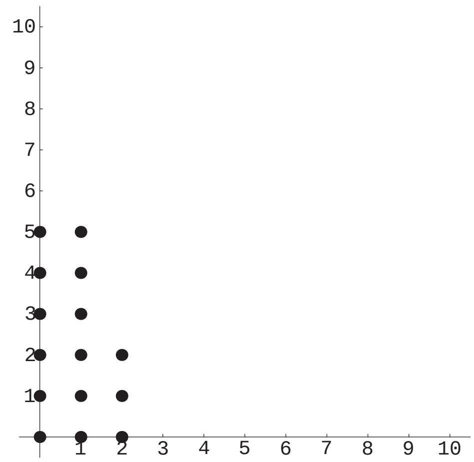

Figure 2.4.3.

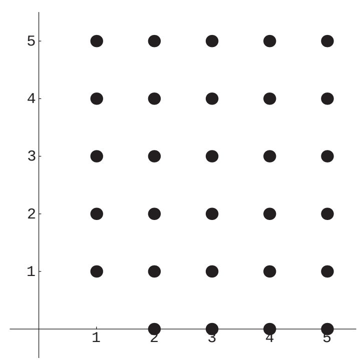

Figure 2.4.4.

following page. Notice that Figure 2.4.4 contains no point of the form $(0, c)$ for any $c \in \mathbb{N}$. As a result, Figure 2.4.5 contains all points of the form $(0, c)$ for all $c \in \mathbb{N}$ and so the number of points in the graph corresponding to R/I is infinite.

$\Diamond$

It would seem to be that case that in order for the number of points in R/I to be finite, then the graph corresponding to the ideal I would have to contain the points $(i, 0)$ and $(0, j)$ for some $i, j \in \mathbb{N}$. What this literally means is that $x^i, y^j \in I$ for some $i, j \in \mathbb{N}$. This turns out to be true and we will prove it in full generality in Chapter 4.

Also notice that if the point (a, b) is contained in the graph of R/I, then every point to the left and below (a, b) in $\mathbb{N} \times \mathbb{N}$ is in the contained in the graph of R/I. That is, every point $(a - c, b - d)$ is in R/I for $0 \leq c \leq a$ and $0 \leq d \leq b$. What this means is that if $\overline{x^a y^b} \in R/I$, then $\overline{x^c y^d} \in R/I$ for $0 \leq c \leq a$ and $0 \leq d \leq b$. We will prove this in Chapter 4, as well.

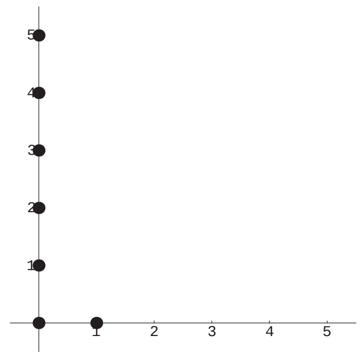

Figure 2.4.5.

These graphs can be very helpful when trying to investigate the behavior of monomial ideals in $k[x, y]$. We will use them again in the next chapter.

3

All Things Hilbert

In this chapter, we define the Hilbert function of a standard graded k-algebra. We will then use this function to create the Hilbert sequence and Hilbert series of the algebra. These sequences and series often have useful properties and we will compute some examples. In the next chapter, we will be solely concerned with generalizing the Hilbert function for certain types of ideals.

3.1 Hilbert Functions, Hilbert Sequences and Hilbert Series

Throughout the remainder of this project, we will denote the dimension of A_n as a vector space over k by $\dim_k(A_n)$.

Proposition 3.1.1. *Let $A = \oplus_{n\geq 0} A_n$ be a finitely generated k-algebra. Then $dim_k(A_n)$ is finite for all $n \geq 0$.*

Proof. Since A is finitely generated, there exists a finite set of homogeneous elements $\{a_i\}_{1\leq i\leq m}$ such that A is spanned as a vector space over k by the monomials $\{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} \mid \alpha_i \in \mathbb{N}\}$. Also, $A_n = \operatorname{span}\{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} \mid \sum_{i=i}^{m} \alpha_i = n\}$. Thus $\dim_k(A_n)$ is the number of monomials of this type. Since the number of a_i's is finite and the number of choices for each α_i is finite, we conclude that $\dim_k(A_n)$ is also finite. □

Definition. Let $A = \oplus_{n\geq 0} A_n$ be a finitely generated graded k-algebra. We define the **Hilbert function** of A by

$$\mathcal{H}(A, n) = \dim_k(A_n).$$

The **Hilbert sequence** of A is the vector $\mathcal{H}_{\mathbf{A}} = (H_0, H_1, H_2, \ldots)$ where $H_i = \mathcal{H}(A, i)$ for all $i \geq 0$. △

Example 3.1.2. Let R and I be as in Example 2.3.8. By Proposition 2.2.6 we have $\mathcal{H}(R,n) = \dim_k(A_n) = \binom{2+n-1}{n} = \binom{n+1}{n} = n+1$. Then $\mathcal{H}_{\mathbf{R}} = (1,2,3,4,5,6,\ldots)$. By counting the number of monomials it takes to span I in Example 2.3.8, we see that $\mathcal{H}(I,0) = \dim_k(I_0) = \mathcal{H}(I,1) = \dim_k(I_1) = 0$ and $\mathcal{H}(I,n) = \dim_k(I_n) = n+1$ when $n \geq 2$ so $\mathcal{H}_{\mathbf{I}} = (0,0,2,3,4,5,\ldots)$. We can repeat the same process with R/I and we discover that $\mathcal{H}(R/I,0) = \dim_k(R_0/I_0) = 1$, $\mathcal{H}(R/I,1) = \dim_k(R_1/I_1) = 2$ and $\mathcal{H}(R/I,n) = \dim_k(R_n/I_n) = 1$ when $n \geq 2$. Hence $\mathcal{H}_{\mathbf{R/I}} = (1,2,1,1,1,1,\ldots)$. ♢

Example 3.1.3. Let R and I be as in Example 2.3.7. We can compute our Hilbert sequences as we did in Example 3.1.2 and we find that

$$\begin{aligned} \mathcal{H}_{\mathbf{R}} &= (1,2,3,4,5,6,7,8,9,10,\ldots), \\ \mathcal{H}_{\mathbf{I}} &= (0,0,0,1,2,4,6,8,9,10,\ldots), \end{aligned}$$

and

$$\mathcal{H}_{\mathbf{R/I}} = (1,2,3,3,3,2,1,0,0,0,\ldots).$$

Note that $\mathcal{H}_{\mathbf{R/I}} = \mathcal{H}_{\mathbf{R}} - \mathcal{H}_{\mathbf{I}}$ when we treat$\mathcal{H}_{\mathbf{R/I}}$, $\mathcal{H}_{\mathbf{R}}$ and $\mathcal{H}_{\mathbf{I}}$ as vectors. this is true in general and we will prove it in Theorem 3.1.6. ♢

First, we will use Hilbert functions to create a generating function.

Definition. Let $A = \oplus_{n\geq 0} A_n$ be a finitely generated graded k-algebra. The **Hilbert series** of A, denoted $\mathcal{F}(A,t)$, is the generating function

$$\mathcal{F}(\mathbf{A},\mathbf{t}) = \sum_{n=0}^{\infty} \mathcal{H}(A,n)t^n.$$

△

Example 3.1.4. Let R and I be as in Examples 2.3.8 and 3.1.2. Then

$$\begin{aligned} \mathcal{F}(R,t) &= 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \cdots \\ &= \frac{d}{dt}\Big(1 + t + t^2 + t^3 + t^4 + \cdots\Big) \\ &= \frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}\ , \end{aligned}$$

$$\begin{aligned} \mathcal{F}(I,t) &= 2t^2 + 3t^3 + 4t^4 + 5t^5 + \cdots \\ &= t\big(2t + 3t^2 + 4t^3 + 5t^4 + \cdots\big) \\ &= t\big(1 + 2t + 3t^2 + 4t^3 + \cdots - 1\big) = t\big(\mathcal{F}(R,t) - 1\big) \\ &= t\left(\frac{1}{(1-t)^2} - 1\right) = \frac{2t^2 - t^3}{(1-t)^2} \end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(R/I,t) &= 1+2t+t^2+t^3+t^4+t^5+\cdots\\ &= 1+2t+t^2\big(1+t+t^2+t^3+t^4+t^5+\cdots\big)\\ &= 1+2t+t^2\left(\frac{1}{1-t}\right)=\frac{1+t-t^2}{1-t}=\frac{1-2t^2+t^3}{(1-t)^2}.\end{aligned}$$

◊

Example 3.1.5. Now let R and I be as in Example 2.3.7. Then we have

$$\mathcal{F}(R,t)=1+2t+3t^2+4t^3+5t^4+6t^5+\cdots=\frac{1}{(1-t)^2}$$

as we saw in Example 3.1.4, and

$$\begin{aligned}\mathcal{F}(I,t) &= 0+0t+0t^2+t^3+2t^4+4t^5+6t^6+8t^7+9t^8+10t^9+\cdots\\ &= t^3+2t^4+4t^5+6t^6+\big(\mathcal{F}(R,t)-1-2t-3t^2-4t^3-5t^4-6t^5-7t^6\big)\\ &= t^3+2t^4+4t^5+6t^6+\left(\frac{1}{(1-t)^2}-1-2t-3t^2-4t^3-5t^4-6t^5-7t^6\right).\end{aligned}$$

Through some routine algebraic manipulation, this can be written as

$$\mathcal{F}(I,t)=\frac{t^3+t^5-t^8}{(1-t)^2}.$$

Also

$$\mathcal{F}(R/I,t)=1+2t+t^2+3t^3+3^4+2t^5+t^6$$

which happens to be finite. If we want to, we can write this as the rational function

$$\mathcal{F}(R/I,t)=\frac{1-t^5-t^3+t^8}{(1-t)^2}.$$

It turns out that $\mathcal{F}(A,t)$ always has the form of a rational function with integer coefficients when A is R, I or R/I and $R=k[x_1,\ldots,x_n]$. Theorem 3.1.9 will address this fact. ◊

Recall that in Examples 3.1.2 and 3.1.3, we saw that $\mathcal{H}_{\mathbf{R/I}}=\mathcal{H}_{\mathbf{R}}-\mathcal{H}_{\mathbf{I}}$. Additionally, observe that in Examples 3.1.4 and 3.1.5, we have $\mathcal{F}(R/I,t)=\mathcal{F}(R,t)-\mathcal{F}(I,t)$. In fact, this is true in general as we will now prove.

Theorem 3.1.6. *Let $A=\oplus_{n\geq 0}A_n$ be a graded k-algebra with a graded ideal $I=\oplus_{n\geq 0}I_n$. Then $\mathcal{H}_{\mathbf{R/I}}=\mathcal{H}_{\mathbf{R}}-\mathcal{H}_{\mathbf{I}}$.*

Proof. By definition $\mathcal{H}_{\mathbf{R/I}} = (H_0, H_1, H_2, \ldots)$ where $H_i = \mathcal{H}(R/I, i)$ for all $i \geq 0$. Similarly, $\mathcal{H}_{\mathbf{R}} = (A_0, A_1, A_2, \ldots)$ where $A_i = \mathcal{H}(R, i)$ and $\mathcal{H}_{\mathbf{I}} = (B_0, B_1, B_2, \ldots)$ where $B_i = \mathcal{H}(I, i)$ for all $i \geq 0$. Clearly $\mathcal{H}_{\mathbf{R}} - \mathcal{H}_{\mathbf{I}} = (A_0 - B_0, A_1 - B_1, A_2 - B_2, \ldots)$. Thus we only need to show that $H_i = A_i - B_i$ for all $i \geq 0$. Consider H_m, A_m and B_m for some $m \geq 0$. We have $H_m = \mathcal{H}(R/I, m) = \dim_k((R/I)_m) = \dim_k(R_m/I_m)$. Similarly, $A_m = \mathcal{H}(R, m) = \dim_k(R_m)$ and $B_m = \mathcal{H}(I, m) = \dim_k(I_m)$. Since R_m is a vector space over the field k and $I_m \subseteq R_m$, by Corollary 2.1.7 we have $\dim_k(R_m/I_m) = \dim_k(R_m) - \dim_k(I_m)$ and thus $H_m = A_m - B_m$. □

Corollary 3.1.7. *Let $A = \oplus_{n\geq 0} A_n$ be a graded k-algebra with a graded ideal $I = \oplus_{n\geq 0} I_n$. Then $\mathcal{F}(R/I, t) = \mathcal{F}(R, t) - \mathcal{F}(I, t)$.*

Proof. Define the addition of generating functions as follows:

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{j=0}^{\infty} b_j x^j = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

(for a more thorough treatment of generating functions, see [2]).

Then

$$\mathcal{F}(R, t) - \mathcal{F}(I, t) = \sum_{n=0}^{\infty} \mathcal{H}(R, n) t^n - \sum_{n=0}^{\infty} \mathcal{H}(I, n) t^n = \sum_{n=0}^{\infty} (\mathcal{H}(R, n) - \mathcal{H}(I, n)) t^n$$

By Theorem 3.1.6, we have

$$\sum_{n=0}^{\infty} (\mathcal{H}(R, n) - \mathcal{H}(I, n)) t^n = \sum_{n=0}^{\infty} \mathcal{H}(R/I, n) t^n = \mathcal{F}(R/I, t)$$

□

We also observe some other properties of the Hilbert series, particularly when $R = k[x_1, \ldots, x_n]$.

Theorem 3.1.8. *Let $R = k[x_1, \ldots, x_n]$ where deg$(x_i) = 1$ for $1 \leq i \leq n$. Then*

$$\mathcal{F}(R, t) = \frac{1}{(1-t)^n}.$$

Proof. We will prove this by induction on n. First, let $n = 1$. Then $R = k[x] = \oplus_{i\geq 0} R_i$ where $R_i = \text{span}\{x^i\}$. Clearly $\mathcal{H}(A, i) = \dim_k(R_i) = 1$ for all $i \geq 0$. Thus

$$\mathcal{F}(R, t) = \sum_{n=0}^{\infty} \mathcal{H}(A, n) t^n = \sum_{n=0}^{\infty} 1 \cdot t^n = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

Now suppose the result holds for $n > 1$. Then if $R = k[x_1, x_2, \ldots, x_n]$, Theorem 2.2.6 tells us that $\mathcal{H}(R, i) = \dim_k(R_i) = \binom{n+i-1}{i}$ for all $i \in \mathbb{N}$, so by hypothesis

$$\mathcal{F}(R,t) = \binom{n+0-1}{0} + \binom{n+1-1}{1}t + \binom{n+2-1}{2}t^2 + \binom{n+3-1}{3}t^3 + \cdots$$
$$= \binom{n-1}{0} + \binom{n}{1}t + \binom{n+1}{2}t^2 + \binom{n+2}{3}t^3 + \cdots = \frac{1}{(1-t)^n}.$$

Let $R' = k[x_1, \ldots, x_n, x_{n+1}]$. Then $R' = \oplus_{j \geq 0} R'_j$ where

$$\mathcal{H}(R', i) = \dim_k(R'_i) = \binom{(n+1)+i-1}{i} = \binom{n+i}{i}$$

for all $i \in \mathbb{N}$. Hence

$$\mathcal{F}(R',t) = \binom{n+0}{0} + \binom{n+1}{1}t + \binom{n+2}{2}t^2 + \binom{n+3}{3}t^3 + \cdots .$$

If we use the combinatorial identity $\binom{n+i}{i} = \binom{n+i-1}{i} + \binom{n+i-1}{i-1}$ for $0 < i < n+i$ and the fact that $\binom{n}{i} = 0$ when $i < 0$ (see [2, Chapter 2, Section 3]), then

$$\mathcal{F}(R',t) = \left[\binom{n+0-1}{0} + \binom{n+0-1}{0-1}\right] + \left[\binom{n+1-1}{1} + \binom{n+1-1}{1-1}\right]t +$$
$$\left[\binom{n+2-1}{2} + \binom{n+2-1}{2-1}\right]t^2 + \left[\binom{n+3-1}{3} + \binom{n+3-1}{3-1}\right]t^3 + \cdots$$

$$= \left[\binom{n-1}{0} + \binom{n-1}{-1}\right] + \left[\binom{n}{1} + \binom{n}{0}\right]t +$$
$$\left[\binom{n+1}{2} + \binom{n+1}{1}\right]t^2 + \left[\binom{n+2}{3} + \binom{n+2}{2}\right]t^3 + \cdots$$
$$= \left[\binom{n-1}{0} + 0\right] + \binom{n}{1}t + \binom{n}{0}t + \binom{n+1}{2}t^2 + \binom{n+1}{1}t^2 + \binom{n+2}{3}t^3 + \binom{n+2}{2}t^3 + \cdots$$
$$= \left[\binom{n-1}{0} + \binom{n}{1}t + \binom{n+1}{2}t^2 + \binom{n+2}{3}t^3 + \cdots\right] + \left[\binom{n}{0}t + \binom{n+1}{1}t^2 + \binom{n+2}{2}t^3 + \cdots\right]$$
$$= \left[\binom{n-1}{0} + \binom{n}{1}t + \binom{n+1}{2}t^2 + \binom{n+2}{3}t^3 + \cdots\right] + t\left[\binom{n}{0} + \binom{n+1}{1}t + \binom{n+2}{2}t^2 + \cdots\right]$$
$$= \mathcal{F}(R,t) + t\mathcal{F}(R',t).$$

We see that $\mathcal{F}(R',t) = \mathcal{F}(R,t) + t\mathcal{F}(R',t)$ and so by hypothesis $\mathcal{F}(R',t) = \frac{1}{(1-t)^n} + t\mathcal{F}(R',t)$. Then $\mathcal{F}(R',t) - t\mathcal{F}(R',t) = \frac{1}{(1-t)^n}$ which implies that $\mathcal{F}(R',t)(1-t) = \frac{1}{(1-t)^n}$. Thus $\mathcal{F}(R',t) = \frac{1}{(1-t)^{n+1}}$ as was desired. □

It is worth noting that if $R = k[x_1, \ldots, x_n]$ with $\deg(x_i) = e_i > 0$ for $1 \leq i \leq n$ then we have

$$\mathcal{F}(R, t) = \frac{1}{\prod_{i=1}^{n}(1 - t^{e_i})}.$$

Since we are only concerned with standard graded k-algebras, this result is more general than we need it to be, but see [6] for more information.

Theorem 3.1.9. *Let $A = \oplus_{n\geq 0} A_n$ be a finitely generated graded k-algebra. Suppose A is generated by the set of monomials $\{x_i\}_{1\leq i\leq s}$ where $deg(x_i) = e_i$. Then $\mathcal{F}(A, t)$ is a rational function of the form*

$$\frac{f(t)}{\prod_{i=1}^{s}(1 - t^{e_i})}$$

where $f(t) \in \mathbb{Z}[t]$.

Proof. See [1, Theorem 11.1]. □

Notice that all of our previous examples are consistent with this theorem.

3.2 A Continuation of Section 2.4

At the end of the Chapter 2, we created graphs in $\mathbb{N} \times \mathbb{N}$ that corresponded to ideals I in $R = k[x, y]$ and R/I. These same graphs can be used to compute $\mathcal{H}(I, i)$ and $\mathcal{H}(R/I, i)$. Recall from our previous examples that $\mathcal{H}(R, i)$ is simply the number of monomials in x and y of total degree i, and that there are $i + 1$ such monomials. If we were to make a graph for R as we did for I and R/I, we would simply plot every point in $\mathbb{N} \times \mathbb{N}$. Notice that every monomial of total degree i takes the form $x^a y^b$ where $a+b = i$ and thus every point in this graph that corresponds with a monomial of total degree i graph falls on the line $y = i-x$. This is shown in Figure 3.2.1. Keep in mind that the only lines we can see in full are the lines $y = i - x$ for $0 \leq i \leq 10$.

Also note that there are $i + 1$ points on each line $y = i - x$. Hence the number of points that fall on the line $y = i - x$ equals $\dim_k(A_i) = \mathcal{H}(R, i)$. Now to compute $\mathcal{H}(I, i)$, we simply need to make a graph as outlined in Section 2.4 and count the number of points in the graph that fall on the line $y = i - x$.

Example 3.2.1. Let $I = \langle x^3, x^2y^3, y^6\rangle$ as in Examples 2.3.7, 2.4.2, 3.1.3, and 3.1.5. The graphs that correspond with I and R/I are shown in Figures 3.2.2 and 3.2.3, respectively, with the lines $y = i - x$ included for $0 \leq i \leq 20$. We can easily see that

$$\mathcal{H}(I, 0) = \mathcal{H}(I, 1) = \mathcal{H}(I, 2) = 0$$

since there are no points in Figure 3.2.2 that fall on the lines $y = -x$, $y = 1-x$ and $y = 2 - x$. However, there is one point on the line $y = 3 - x$ so we conclude that

$$\mathcal{H}(I, 3) = 1.$$

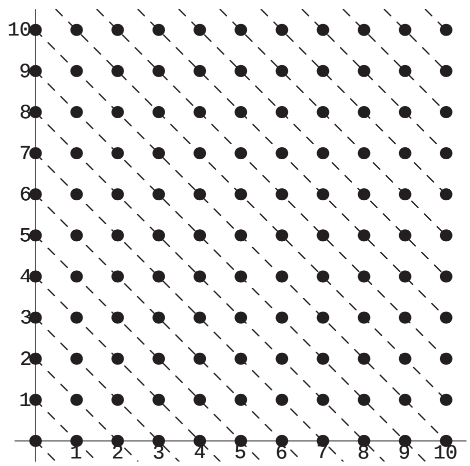

Figure 3.2.1.

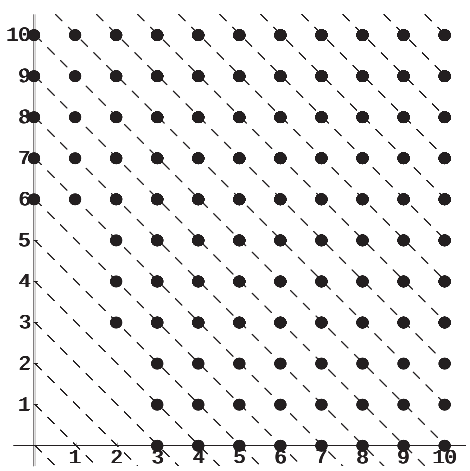

Figure 3.2.2.

Similarly, we can see that

$$\mathcal{H}(I,4) = 2, \mathcal{H}(I,5) = 4, \mathcal{H}(I,6) = 6$$

and

$$\mathcal{H}(I,7+r) = 7+r+1 = 8+r$$

for $r \geq 0$. Hence $\mathcal{H}_{\mathbf{I}} = (0,0,0,1,2,4,6,8,9,10,\ldots)$. We can proceed in a similar manner to see that $\mathcal{H}(R/I,0) = 1$, $\mathcal{H}(R/I,1) = 2$, $\mathcal{H}(R/I,2) = 3$, $\mathcal{H}(R/I,3) = 3$, $\mathcal{H}(R/I,4) = 3$, $\mathcal{H}(R/I,5) = 2$, $\mathcal{H}(R/I,6) = 1$, and $\mathcal{H}(R/I,7+r) = 0$ for $r \geq 0$. Hence $\mathcal{H}_{\mathbf{R/I}} = (1,2,3,3,3,2,1,0,0,0,\ldots)$. Both $\mathcal{H}_{\mathbf{I}}$ and $\mathcal{H}_{\mathbf{R/I}}$ match the sequences we obtained in Example 3.1.3. $\Diamond$

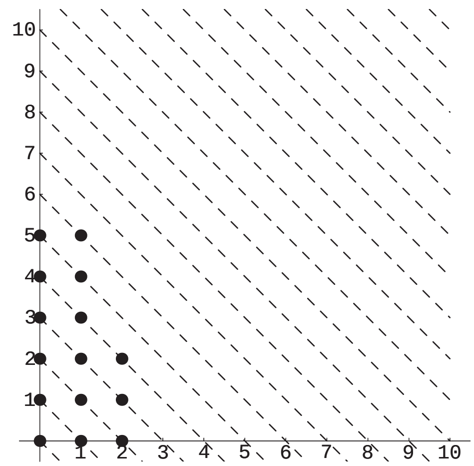

Figure 3.2.3.

Example 3.2.2. Let $I = \langle x^2, xy \rangle$ as in Examples 2.3.8, 2.4.4, 3.1.2, and 3.1.4. The graphs that correspond with I and R/I are shown in Figures 3.2.4 and 3.2.5, respectively, with the lines $y = i - x$ included for $0 \leq i \leq 10$.

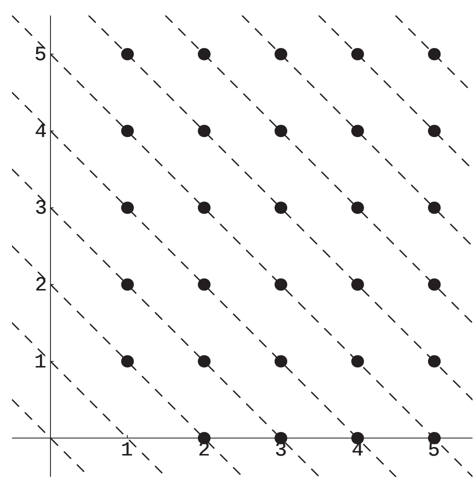

Figure 3.2.4.

We can easily see that $\mathcal{H}_{\mathbf{I}} = (0, 0, 2, 3, 4, 5, \ldots)$ and $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 1, 1, 1, 1, \ldots)$ as we found in Example 3.1.2. $\Diamond$

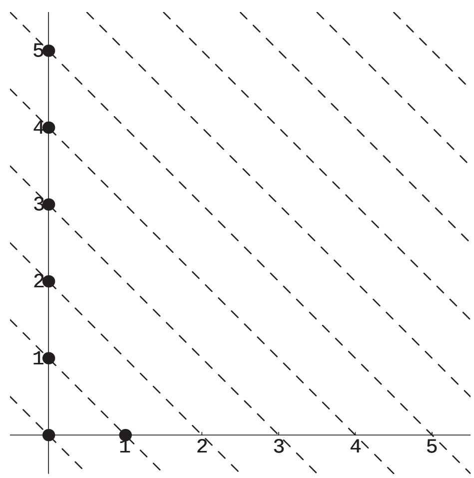

Figure 3.2.5.

4

Hilbert Sequences of Monomial Ideals

In Chapters 2 and 3, we have computed several examples using monomial ideals. However, as we have yet to define a monomial ideal, there is a bit more theory that needs to be addressed before we can continue our exploration into the different types of Hilbert sequences in $k[x_1, \ldots, x_n]$. In this chapter, we will finally define a monomial ideal and will discuss some important attributes of this type of ideal. Then in Sections 4.3, 4.4 and 4.5, I will present my own results which include some basic properties of Hilbert functions and Hilbert sequences of monomial ideals in $k[x_1, \ldots, x_n]$ and a classification of all possible Hilbert sequences for R/I when $R = k[x]$ or $k[x, y]$ and I is any monomial ideal in R.

4.1 Monomial Ideals

Recall from Section 2.2 that if $R = k[x_1, \ldots, x_n]$, then a monomial in R is a product of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha_i \in \mathbb{N}$ for $1 \leq i \leq n$ which we denote by $\mathbf{x}^\alpha$. Note that $x_1^0 x_2^0 \cdots x_n^0 = 1$ is always a monomial. Also recall that the **total degree** of $\mathbf{x}^\alpha$ is $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \sum_{i=1}^n \alpha_i$ and we write $\sum_{i=1}^n \alpha_i = |\alpha|$. We will be using this notation extensively throughout this chapter.

Example 4.1.1. It is often helpful to think of α as a vector with the vector properties of addition and subtraction (after all, it is an n-tuple of non-negative integers). In fact, addition and subtraction correspond directly to the multiplication and division of monomials. For example, if $R = k[x_1, x_2, x_3, x_4]$ then $m_1 = {x_1}^7 {x_2}^2 {x_3}^4 {x_4}^5$ and $m_2 = x_1 {x_2}^2 {x_4}^3 = x_1 {x_2}^2 {x_3}^0 {x_4}^3$ are both monomials in R. To utilize our shorthand notation, we write $m_1 = \mathbf{x}^\alpha = \mathbf{x}^{(7,2,4,5)}$ and $m_2 = \mathbf{x}^\beta = \mathbf{x}^{(1,2,0,3)}$ so $\alpha = (7, 2, 4, 5)$

and $\beta = (1, 2, 0, 3)$ with $|\alpha| = 18$ and $|\beta| = 6$. Hence

$$\begin{aligned} m_1 m_2 &= {x_1}^7 {x_2}^2 {x_3}^4 {x_4}^5 x_1 {x_2}^2 {x_3}^0 {x_4}^3 \\ &= ({x_1}^7 x_1)({x_2}^2 {x_2}^2)({x_3}^4 {x_3}^0)({x_4}^5 {x_4}^3) = {x_1}^{7+1} {x_2}^{2+2} {x_3}^{4+0} {x_4}^{5+3} \\ &= {x_1}^8 {x_2}^4 {x_3}^4 {x_4}^8 \end{aligned}$$

by an algebraic rule of exponents. Thus $\mathbf{x}^\alpha \mathbf{x}^\beta = \mathbf{x}^\gamma$ where $\gamma = (8, 4, 4, 8)$. Note that $\alpha + \beta = \gamma$ and $|\gamma| = 24 = |\alpha| + |\beta|$.

Similarly,

$$\frac{m_1}{m_2} = \frac{{x_1}^7 {x_2}^2 {x_3}^4 {x_4}^5}{x_1 {x_2}^2 {x_3}^0 {x_4}^3} = {x_1}^{7-1} {x_2}^{2-2} {x_3}^{4-0} {x_4}^{5-3} = {x_1}^6 {x_2}^0 {x_3}^4 {x_4}^2 = {x_1}^6 {x_3}^4 {x_4}^2$$

and so $\frac{\mathbf{x}^\alpha}{\mathbf{x}^\beta} = \mathbf{x}^\delta$ where $\delta = (6, 0, 4, 2)$. Note that $\alpha - \beta = \delta$ and $|\delta| = 12 = |\alpha| - |\beta|$. $\lozenge$

It is not difficult to show that for all $\mathbf{x}^\alpha, \mathbf{x}^\beta \in k[x_1, \ldots, x_n]$, then $\mathbf{x}^\alpha \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta}$ and the total degree of $\mathbf{x}^\alpha \mathbf{x}^\beta$ is $|\alpha| + |\beta|$. Similarly, if $\alpha_i \geq \beta_i$ for $1 \leq i \leq n$, then $\frac{\mathbf{x}^\alpha}{\mathbf{x}^\beta} = \mathbf{x}^{\alpha-\beta}$ and the total degree of $\frac{\mathbf{x}^\alpha}{\mathbf{x}^\beta}$ is $|\alpha| - |\beta|$.

We will now define a monomial ideal. This definition is hardly surprising.

Definition. An ideal I of $R = k[x_1, \ldots, x_n]$ is called a **monomial ideal** if there exists A, a set of n-tuples of non-negative integers (possibly an infinite set), such that $I = \langle \mathbf{x}^\alpha \mid \alpha \in A \rangle$. $\triangle$

Therefore, an ideal is a monomial ideal if it is generated by a set of monomials as we have previously assumed. It is important to note that since every monomial of total degree i is homogeneous of degree i, then Proposition 2.3.5 tells us that every monomial ideal is a graded ideal.

Example 4.1.2. Let $R = k[x, y]$ and let $A = \{(3, 0), (2, 3), (0, 6), (8, 1)\}$. Then $I = \langle \mathbf{x}^{(3,0)}, \mathbf{x}^{(2,3)}, \mathbf{x}^{(0,6)}, \mathbf{x}^{(8,1)} \rangle = \langle x^3 y^0, x^2 y^3, x^0 y^6, x^8 y^1 \rangle = \langle x^3, x^2 y^3, y^6, x^8 y \rangle$ is a monomial ideal. We saw this ideal in Example 2.4.2. Notice that when we made the graph in $\mathbb{N} \times \mathbb{N}$ corresponding to I, we plotted the point $\alpha = (a, b)$ and all of the points to the right and above (a, b) for each $\mathbf{x}^\alpha \in I$. Thus we plot each point $(\alpha_1 + c, \alpha_2 + d)$ such that $(\alpha_1, \alpha_2) \in A$ and $c, d \in \mathbb{N}$ in $\mathbb{N} \times \mathbb{N}$. $\lozenge$

4.2 Some Properties of Monomials and Monomial Ideals

Proposition 4.2.1. *Let $R = k[x_1, \ldots, x_n]$ and let I be a monomial ideal of R. Let m be a monomial in R such that $m \notin I$. For each equivalence class $\overline{m} = m + I \in R/I$, m is the only monomial in that class.*

Proof. Let m_1 and m_2 be monomials such that $m_1 \notin I$ and $m_1 \neq m_2$. Suppose $m_1 + I = m_2 + I$. Then $(m_1 - m_2) + I = 0 + I = I$. By Proposition 2.1.5, this implies that $m_1 - m_2 \in I$. Since I is a monomial ideal, it is also a graded ideal. By Proposition 2.3.2, $m_1 \in I$ and $m_2 \in I$, a contradiction. $\square$

Proposition 4.2.1 shows us that it is reasonable to abuse notation and write $m \in R/I$ instead of $\overline{m}$ when I is a monomial ideal and m is a monomial. We will do this throughout the remainder of this project. In particular, for all monomials $m \in R$, either $m \in I$ or m is the unique monomial in the equivalence class $m + I = \overline{m}$ and so we write $m \in R/I$. Recall that Corollary 2.1.7 tells us that $\dim_k (R_i/I_i) = \dim_k(R_i) - \dim_k(I_i)$. Since R_i is the span of all monomials of total degree i and I_i is the span of all monomials of total degree i in R_i that are also in I, then it makes sense to think of R_i/I_i as the span of all monomials of total degree i in R_i that are not in I_i.

Lemma 4.2.2. *Let $I = \langle \mathbf{x}^\alpha \mid \alpha \in A\rangle$ be a monomial ideal and let $\mathbf{x}^\beta$ be a monomial in $R = k[x_1, \ldots, x_n]$. Then $\mathbf{x}^\beta \in I$ if and only if $\mathbf{x}^\beta$ is divisible by $\mathbf{x}^\alpha$ for some $\alpha \in A$.*

Proof. The proof of this is very straightforward and I refer the reader to [3, Chapter 2, §4, Lemma 2]. □

Note that $\mathbf{x}^\alpha | \mathbf{x}^\beta$ if and only if there exists some $\gamma \in \mathbb{N}^n$ such that $\mathbf{x}^\alpha \mathbf{x}^\gamma = \mathbf{x}^{\alpha+\gamma} = \mathbf{x}^\beta$. Thus $\mathbf{x}^\alpha | \mathbf{x}^\beta$ if there exists some $\gamma \in \mathbb{N}^n$ such that $\alpha + \gamma = \beta$. Hence the set $\{\alpha + \gamma \mid \gamma \in \mathbb{N}^n\}$ is the set of all exponents of monomials divisible by $\mathbf{x}^\alpha$. We conclude that if $\mathbf{x}^\alpha \in I$, then $\mathbf{x}^{\alpha+\gamma} \in I$ for all $\gamma \in \mathbb{N}^n$. We have intuitively been using this idea throughout this project, but now we can formally see why it is true.

Corollary 4.2.3. *Let $R = k[x_1, \ldots, x_n]$ and suppose I is a monomial ideal in R. If $\mathbf{x}^\beta \in R/I$ and $\mathbf{x}^\alpha | \mathbf{x}^\beta$, then $\mathbf{x}^\alpha \in R/I$ as well.*

Proof. Suppose $\mathbf{x}^\beta \in R/I$ and $\mathbf{x}^\alpha | \mathbf{x}^\beta$ but $\mathbf{x}^\alpha \in I$. By Lemma 4.2.2, there exists $\gamma \in \mathbb{N}$ such that $\mathbf{x}^\alpha \mathbf{x}^\gamma = \mathbf{x}^\beta$. By the property of ideals, $\mathbf{x}^\beta = \mathbf{x}^\alpha \mathbf{x}^\gamma \in I$, a contradiction. Thus $\mathbf{x}^\alpha \in R/I$. □

The most important use of this corollary is that if we know that there exists a non-zero element $\mathbf{x}^\beta \in R/I$, we can write $\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, and we can conclude that $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \in R/I$ when $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$.

Example 4.2.4. Let $R = k[x, y, z]$. Suppose that $xyz = x^1y^1z^1 \in I$. Then by Lemma 4.2.2, $(xyz)(x^ay^bz^c) = x^{a+1}y^{b+1}z^{c+1} \in I$ for all $a, b, c \in \mathbb{N}$. Thus I contains all monomials of the form $x^iy^jz^k$ for $i, j, k \geq 1$.

Now suppose $xyz = x^1y^1z^1 \in R/I$ instead. Since xyz is a non-zero element, by Corollary 4.2.3, $x^1y^1z^0 = xy$, $x^1y^0z^1 = xz$, $x^0y^1z^1 = yz$, $x^1y^0z^0 = x$, $x^0y^1z^0 = y$, $x^0y^0z^1 = z$, and $x^0y^0z^0 = 1$ are also in R/I. When we are investigating a monomial ideal I, if $\mathbf{x}^\alpha \in I$ then we know that every monomial multiple of $\mathbf{x}^\alpha$ is also in I. Conversely, if $\mathbf{x}^\beta \in R/I$, then we know that every monomial divisor of $\mathbf{x}^\beta$ is also in R/I. ◊

We now consider the case where I is generated by the set $\{\mathbf{x}^{\alpha(1)}, \mathbf{x}^{\alpha(2)}, \ldots, \mathbf{x}^{\alpha(n)}\}$ where $\alpha(i) = (0, \ldots, 0, e_i, 0, \ldots)$ with $e_i \in \mathbb{N}$ and $0 \leq i \leq n$.

Lemma 4.2.5. *Let $R = k[x_1, \dots, x_n]$ and suppose that $I = \langle x_1^{e_1}, x_2^{e_2}, \dots, x_n^{e_n} \rangle$ where $e_1, e_2, \dots, e_n \in \mathbb{N}$. Then $\mathbf{x}^\beta \in R/I$ if and only if $\beta_i < e_i$ for all $1 \le i \le n$.*

Proof. ($\Rightarrow$) Let $\mathbf{x}^\beta \in R/I$, but suppose there exists some $i \in \{1, \dots, n\}$ such that $\beta_i \ge e_i$. Hence there exists some $r \in \mathbb{N}$ such that $\beta_i = e_i + r$. Then $x_i^{\beta_i} = x_i^{e_i+r} = x_i^{e_i} x_i^r$. By hypothesis, $x_i^{e_i} \in I$ and so by the property of ideals, we have $x_i^{\beta_i} = x_i^{e_i} x_i^r \in I$.

If $\mathbf{x}^\beta = \mathbf{x}^{(\beta_1, \beta_2, \dots, \beta_{i-1}, 0, \beta_{i+1}, \dots, \beta_n)} x_i^{\beta_i}$, then since $x_i^{\beta_i} \in I$, we conclude that $\mathbf{x}^\beta \in I$ as well, a contradiction. Thus there exists no such $i \in \{1, \dots, n\}$ such that $\beta_i \ge e_i$. Hence $\beta_i < e_i$ for $0 \le i \le n$.

($\Leftarrow$) Let $\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ with $\beta_i < e_i$ for $1 \le i \le n$. Thus $\mathbf{x}^\beta$ is not divisible by any monomial in I and so by Lemma 4.2.2, we have $\mathbf{x}^\beta \notin I$. Therefore, $\mathbf{x}^\beta \in R/I$. ☐

Notice that this lemma tells us that $\beta_i \le e_i - 1$ for $1 \le i \le n$ and so the monomial in R/I with the largest possible total degree is $x_1^{e_1-1} x_2^{e_2-1} \cdots x_n^{e_n-1}$.

Corollary 4.2.6. *Let $R = k[x_1, \dots, x_n]$ and suppose that $I = \langle x_1^{e_1}, x_2^{e_2}, \dots, x_n^{e_n} \rangle$ with $e_1, e_2, \dots, e_n \in \mathbb{N}$. If $\mathbf{x}^\beta \in R/I$ then $|\beta| \le \left(\sum_{j=1}^n e_j \right) - n$.*

Proof. If $\mathbf{x}^\beta \in R/I$, then by Lemma 4.2.5 we have $\beta_i < e_i$ which implies that $\beta_i \le e_i - 1$ for $1 \le i \le n$. Then

$$|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n \le (e_1 - 1) + (e_2 - 1) + \cdots + (e_n - 1) = \Big(\sum_{j=1}^n e_j \Big) - n.$$

☐

Example 4.2.7. Let $R = k[x, y, z]$. Suppose that $I = \langle x^2, y^3, z^4 \rangle$. To figure out which monomials are in R/I, we begin by finding $\mathbf{x}^\beta \in R/I$ of maximum possible total degree. By Lemma 4.2.5 and Corollary 4.2.6, $|\beta| = (2 + 3 + 4) - 3 = 6$ and so $\mathbf{x}^\beta = x^{2-1} y^{3-1} z^{4-1} = x^1 y^2 z^3 = xy^2 z^3$. Notice that the set of all monomials in R of total degree 7 is

$$\{x^7, y^7, z^7, x^6 y, x^5 y^2, x^4 y^3, x^3 y^4, x^2 y^5, xy^6, x^6 z, x^5 z^2, x^4 z^3, x^3 z^4, x^2 z^5,$$
$$xz^6, y^6 z, y^5 z^2, y^4 z^3, y^3 z^4, y^2 z^5, yz^6, x^5 yx, x^4 y^2 z, x^4 yz^2, x^3 y^3 z, x^3 y^2 z^2,$$
$$x^3 yz^3, x^2 y^4 z, x^2 y^3 z^2, x^2 y^2 z^3, x^2 yz^4, xy^5 z, xy^4 z^2, xy^3 z^3, xy^2 z^4, xyz^5\}$$

and it is not difficult to check that every monomial in this set is also in I. Thus $\mathbf{x}^\beta$ does indeed have the maximum possible total degree in R/I.

Now by Corollary 4.2.3, we have $x^a y^b z^c \in R/I$ for $0 \le a \le 1$, $0 \le b \le 2$ and $0 \le c \le 3$. Therefore the monomials in R/I are

$$\{1, x, y, z, y^2, z^2, xy, xz, yz, z^3, xy^2, xz^2, y^2 z, yz^2, xyz,$$
$$xz^3, y^2 z^2, yz^3, xy^2 z, xyz^2, y^2 z^3, xyz^3, xy^2 z^2, xy^2 z^3\}.$$

Also, all monomials of the form $x^a y^b z^c$ are in I for $a \geq 2$, $b \geq 3$, or $c \geq 4$. Note that these are the remaining monomials in x, y, z that are not listed immediately above. ♢

We now list several lemmas and theorems that characterize the structure of monomial ideals in $k[x_1, \ldots, x_n]$. All proofs can be found in [3].

Lemma 4.2.8. *Let I be a monomial ideal in $R = k[x_1, \ldots, x_n]$ and let $f \in R$. Then the following are equivalent:*

1. *$f \in I$.*

2. *Every monomial of f lies in I.*

3. *f is a k-linear combination of the monomials in I.*

This lemma tells us that a polynomial f is contained in a monomial ideal if and only if every monomial of f is contained in the ideal. For example, if $R = \mathbb{Z}[x, y, z]$ and $I = \langle x^2, xy^3, y^4z, xyz\rangle$, then $f = 3x^7 + 7xy^3z + 2y^4z + xy^2z^2 \in I$ since $x^7 = (x^2)(x^5)$, $xy^3z = (xy^3)(z) = (xyz)(y^2)$, $y^4z = (y^4z)(1)$, and $xy^2z^2 = (xyz)(yz)$.

Theorem 4.2.9. *(Dickson's Lemma) Every monomial ideal in $k[x_1, \ldots, x_n]$ is generated by a finite number of monomials.*

This theorem is very important and not intuitively obvious. We saw in Section 2.4 and Example 2.4.2 that two different generating sets could generate the same ideal but one ideal contained a redundant generator. The idea behind the proof of Dickson's Lemma is an inductive argument on the number of variables that proves that in any monomial ideal I generated by an infinite set of monomials, we can eliminate redundant generators until we have a finite number of generators that generate the same ideal. These redundant monomials turn out to be monomial multiples of other monomials in the ideal. For example, if $R = k[x, y]$ and $I = \langle x^2, x^2y, x^2y^2, x^2y^3, x^2y^4, x^2y^5, \ldots\rangle$, we see that I can also be written as $I = \langle x^2\rangle$.

Theorem 4.2.10. *(The Hilbert Basis Theorem) Let $R = k[x_1, \ldots, x_n]$. Every ideal in R has a finite generating set. That is, $I = \langle f_1, f_2, \ldots, f_s\rangle$ for some $f_1, f_2, \ldots, f_s \in I$.*

The Hilbert Basis Theorem tells us that any ideal, monomial or otherwise, is finitely generated. There are numerous proofs of this theorem as it is a very important result in algebra. However, in the proof given in [3], much of the work is done by Dickson's Lemma. The fact that this theorem can be proved using monomial ideals shows that these ideals play a very important role in the study of all ideals in $k[x_1, \ldots, x_n]$.

4.3 Several Results in n-dimensions

In this section, we investigate some properties of Hilbert functions in $R = k[x_1, \ldots, x_n]$. We then look at the Hilbert functions for several classes of monomial ideals and use these results to form Hilbert sequences. We will also characterize the ideals I for which the Hilbert sequence of R/I has a finite number of non-zero entries.

The first observation we make is that in every Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$, once the entry zero appears in the sequence, every entry that follows is zero.

Theorem 4.3.1. *Let $R = k[x_1, \ldots, x_n]$ and let I be a monomial ideal in R. If $\mathcal{H}(R/I, i) = 0$ for some $i \in \mathbb{N}$, then $\mathcal{H}(R/I, j) = 0$ for all $j \geq i$.*

Proof. Assume that $\mathcal{H}(R/I, i) = 0$ for some $i \in \mathbb{N}$. This means that all of the monomials of total degree i are in I. Then $\mathbf{x}^\alpha \in I$ whenever $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = i$. Now let $\mathbf{x}^\beta$ be a monomial of total degree $i+1$. Then $\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ where $|\beta| = i + 1$. Let β_k be the first non-zero component of β. Then $\mathbf{x}^\beta = x_k^{\beta_k} \cdots x_n^{\beta_n} = x_k(x_k^{\beta_k - 1} x_{k+1}^{\beta_{k+1}} \cdots x_n^{\beta_n}) = x_k \mathbf{x}^\gamma$ where $\mathbf{x}^\gamma = x_k^{\beta_k - 1} x_{k+1}^{\beta_{k+1}} \cdots x_n^{\beta_n}$. Note that $|\gamma| = i$ and so $\mathbf{x}^\gamma \in I$. Therefore, $\mathbf{x}^\beta = x_k \mathbf{x}^\gamma \in I$ also. We conclude that all monomials of total degree $i+1$ are in I and so $\mathcal{H}(R/I, i+1) = 0$. By iterating this procedure, we see that $\mathcal{H}(R/I, j) = 0$ for all $j \geq i$. $\square$

Corollary 4.3.2. *Let $R = k[x_1, \ldots, x_n]$ and let I be a proper monomial ideal in R. If $\mathcal{H}(R/I, j) > 0$ for some $j \in \mathbb{N}$, then $\mathcal{H}(R/I, i) > 0$ for all $i \leq j$.*

Proof. This corollary is the contrapositive of Theorem 4.3.1. $\square$

Intuitively, Theorem 4.3.1 makes sense. For example, if all monomials of total degree i are in I, that means that there are no monomials of total degree i in R/I and so $\mathcal{H}(R/I, i) = 0$. We have seen that to obtain all monomials of total degree $i + j$ for $j \geq 0$, we simply multiply each monomial of total degree i by $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $|\alpha| = j$. Thus every monomial of total degree $i + j$ is a monomial multiple of some monomial of total degree i, which by hypothesis, is an element of I. Hence there are no monomials of total degree $i + j$ in R/I for $j \geq 0$. Similarly, Corollary 4.3.2 makes sense in terms of Corollary 4.2.3 since if a monomial of total degree i is in R/I, then every monomial that divides that monomial is also in R/I.

We now determine the Hilbert function for ideals generated by a single monomial.

Theorem 4.3.3. *Let $I = \langle \mathbf{x}^\alpha \rangle$ in $R = k[x_1, \ldots, x_n]$. Then*

$$\mathcal{H}(R/I, i) = \begin{cases} \binom{i+n-1}{i} & \text{for } 0 \leq i < |\alpha| \\ \binom{i+n-1}{i} - \binom{i-|\alpha|+n-1}{i-|\alpha|} & \text{for } i \geq |\alpha|. \end{cases}$$

Proof. Recall that $\mathcal{H}(R, i) = \binom{i+n-1}{i}$. Clearly $\mathcal{H}(R/I, i) = \binom{i+n-1}{i}$ for $0 \leq i < |\alpha|$ since no monomial of total degree less than $|\alpha|$ can be in I. Let $p = |\alpha| + r$ where

$r \geq 0$. Suppose $\mathbf{x}^\beta$ is a monomial of total degree p. Note that $\mathbf{x}^\beta \in I$ if and only if $\mathbf{x}^\beta = \mathbf{x}^\alpha \mathbf{x}^\gamma$ for some $\mathbf{x}^\gamma \in R$. This implies that $\beta = \alpha + \gamma$ for some $\gamma \in \mathbb{N}^n$. Since $|\beta| = p$ then we conclude that $|\gamma| = r$. To calculate $\mathcal{H}(I, |\alpha| + r)$, we only need to figure out how many ways we can construct $\mathbf{x}^\gamma$. This is the same as counting the number of monomials of total degree r in n variables. By Proposition 2.2.6 this number is $\binom{r+n-1}{r}$. Therefore $\mathcal{H}(R/I, |\alpha| + r) = \binom{|\alpha|+r+n-1}{|\alpha|+r} - \binom{r+n-1}{r}$. More explicitly $\mathcal{H}(R/I, p) = \binom{p+n-1}{p} - \binom{p-|\alpha|+n-1}{p-|\alpha|}$. □

Now that we can calculate the Hilbert functions of monomial ideals of this form, we can use the functions to construct the Hilbert sequences for R/I. Thus we are able to categorize all possible Hilbert sequences of ideals generated by a single monomial.

Example 4.3.4. Let $R = k[x, y]$ and suppose $I = \langle x^3y^2 \rangle$. Using the methods from Chapter 3, we can see that

$$I_0 = I_1 = I_2 = I_3 = I_4 = \{0\}.$$

Also, we have

$$I_5 = \operatorname{span}\{x^3y^2\},$$
$$I_6 = \operatorname{span}\{x^4y^2, x^3y^3\},$$
$$I_7 = \operatorname{span}\{x^5y^2, x^4y^3, x^3y^4\},$$
$$I_8 = \operatorname{span}\{x^6y^2, x^5y^3, x^4y^4, x^3y^5\},$$

and so forth with

$$I_i = \operatorname{span}\{x^{i-2}y^2, x^{i-3}y^3, \ldots, x^3y^{i-3}\}.$$

Thus

$$R_0/I_0 \cong \operatorname{span}\{1\},$$
$$R_1/I_1 \cong \operatorname{span}\{x, y\},$$
$$R_2/I_2 \cong \operatorname{span}\{x^2, xy, y^2\},$$
$$R_3/I_3 \cong \operatorname{span}\{x^3, x^2y, xy^2, y^3\},$$
$$R_4/I_4 \cong \operatorname{span}\{x^4, x^3y, x^2y^2, xy^3, y^4\},$$
$$R_5/I_5 \cong \operatorname{span}\{x^5, x^4y, x^2y^3, xy^4, y^5\},$$
$$R_6/I_6 \cong \operatorname{span}\{x^6, x^5y, x^2y^4, xy^5, y^6\},$$
$$R_7/I_7 \cong \operatorname{span}\{x^7, x^6y, x^2y^5, xy^6, y^7\},$$
$$R_8/I_8 \cong \operatorname{span}\{x^8, x^7y, x^2y^6, xy^7, y^8\},$$

and so forth with

$$R_i/I_i \cong \operatorname{span}\{x^i, x^{i-1}y, x^2y^{i-2}, xy^{i-1}, y^i\}.$$

We see that

$$\begin{aligned}
\mathcal{H}(R/I, 0) &= 1,\\
\mathcal{H}(R/I, 1) &= 2,\\
\mathcal{H}(R/I, 2) &= 3,\\
\mathcal{H}(R/I, 3) &= 4,\\
\mathcal{H}(R/I, 4) &= 5,\\
\mathcal{H}(R/I, 5) &= 5,\\
\mathcal{H}(R/I, 6) &= 5,\\
\mathcal{H}(R/I, 7) &= 5,\\
\mathcal{H}(R/I, 8) &= 5
\end{aligned}$$

and so forth with

$$\mathcal{H}(R/I, 4 + r) = 5$$

for $r \geq 0$. Then $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 4, 5, 5, 5, 5, \ldots)$.

Now let us use Theorem 4.3.3 to compute this example. In this case, we have $I = \langle x^3y^2\rangle = \langle \mathbf{x}^{(3,2)}\rangle$ and $|\alpha| = 3+2 = 5$. When calculating $\mathcal{H}(R/I, i)$ for $0 \leq i < 5$, the first case of Theorem 4.3.3 tells us that

$$\begin{aligned}
\mathcal{H}(R/I, 0) &= \binom{0+2-1}{0} = \binom{1}{0} = 1,\\
\mathcal{H}(R/I, 1) &= \binom{1+2-1}{1} = \binom{2}{1} = 2,\\
\mathcal{H}(R/I, 2) &= \binom{2+2-1}{2} = \binom{3}{2} = 3,\\
\mathcal{H}(R/I, 3) &= \binom{3+2-1}{3} = \binom{4}{3} = 4,
\end{aligned}$$

and finally

$$\mathcal{H}(R/I, 4) = \binom{4+2-1}{4} = \binom{5}{4} = 5.$$

For all $i \geq 5$, the second case of Theorem 4.3.3 tells us that we have

$$\begin{aligned}
\mathcal{H}(R/I, i) &= \binom{i+2-1}{i} - \binom{i-5+2-1}{i-5} = \binom{i+1}{i} - \binom{i-4}{i-5}\\
&= (i+1) - (i-4) = 5.
\end{aligned}$$

Hence $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 4, 5, 5, 5, 5, \ldots)$ which agrees with the result we obtained above. ◊

Example 4.3.5. Let $R = k[x, y, z]$ and suppose $I = \langle z \rangle$. Intuitively, R/I is the span over k of all monomials in x, y, z of the form $x^a y^b z^0 = x^a y^b$. Then R/I is the span of all monomials in x, y over k which is simply $k[x, y]$, so we would expect $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 4, 5, 6, 7, 8, \ldots)$, but let us use Theorem 4.3.3 to check this.

Since $I = \langle z \rangle = \langle \mathbf{x}^{(0,0,1)} \rangle$, we have $|\alpha| = 0{+}0{+}1 = 1$. Thus $\mathcal{H}(R/I, 0) = \binom{0+3-1}{0} = \binom{2}{0} = 1$. Now for all $i \geq 1$, we have

$$\begin{aligned}\mathcal{H}(R/I, i) &= \binom{i+3-1}{i} - \binom{i-1+3-1}{i-1} = \binom{i+2}{i} - \binom{i+1}{i-1} \\ &= \frac{(i+2)(i+1)}{2} - \frac{(i+1)(i)}{2} \\ &= \frac{(i+1)\big((i+2)-i\big)}{2} = \frac{(i+1)(2)}{2} = i+1.\end{aligned}$$

Therefore, $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 4, 5, 6, 7, 8, \ldots)$ as we anticipated. ♢

The surprising consequence of Theorem 4.3.3 is that when I is generated by a single monomial, the Hilbert sequence of R/I will be the same no matter which monomial of a given total degree we choose to generate I. This is because $\mathcal{H}(R/I, i)$ is solely determined by i, the number of variables in $k[x_1, \ldots, x_n]$, and the total degree of the generator of I. For example, $\mathcal{H}_{\mathbf{R}/\langle \mathbf{x^2} \rangle} = \mathcal{H}_{\mathbf{R}/\langle \mathbf{xy} \rangle} = \mathcal{H}_{\mathbf{R}/\langle \mathbf{y^2} \rangle} = (1, 2, 2, 2, 2, 2, \ldots)$.

Now we will determine which Hilbert sequences will have a finite number of non-zero entries. We begin by making a few definitions about this particular kind of sequence.

Definition. Let $A = \oplus_{n \geq 0} A_n$ be a finitely generated standard graded k-algebra. Then $\mathcal{H}_{\mathbf{A}}$ is **finite** if there exists some $i \in \mathbb{N}$ such that $\mathcal{H}(A, j) = 0$ for all $j \geq i$. By Theorem 4.3.1, it is sufficient to find a single i for which $\mathcal{H}(A, i) = 0$. If $\mathcal{H}_{\mathbf{A}}$ is finite and $\mathcal{H}(A, i) = 0$ where i is the least such integer, we write $\mathcal{H}_{\mathbf{A}} = (\mathcal{H}(A, 0), \mathcal{H}(A, 1), \mathcal{H}(A, 3), \ldots, \mathcal{H}(A, i-1))$ and define the **length** of $\mathcal{H}_{\mathbf{A}}$ to be the number of (non-zero) entries in $\mathcal{H}_{\mathbf{A}}$. In this case, the length of $\mathcal{H}_{\mathbf{A}}$ is i. △

Example 4.3.6. When $R = k[x, y]$ and $I = \langle x^3, x^2y^3, y^6 \rangle$, we saw in Example 3.1.3 that $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 3, 3, 2, 1, 0, 0, 0 \ldots)$ and so $\mathcal{H}_{\mathbf{R/I}}$ is finite. We write $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 3, 3, 2, 1)$ and so $\mathcal{H}_{\mathbf{R/I}}$ has length 7. ♢

Example 4.3.7. Let $R = k[x, y, z]$ and $I = \langle x^2, y^3, z^4 \rangle$. It follows from Example 4.2.7 that $\mathcal{H}_{\mathbf{R/I}} = (1, 3, 5, 6, 5, 3, 1)$ and has length 7. ♢

We will next consider ideals in $k[x_1, \ldots, x_n]$ generated only by monomials of the form $x_i^{e_i}$ for $0 \leq i \leq n$ as in Example 4.3.7. Ideals of this type have very interesting Hilbert sequences. We have already worked with this type of ideal in Lemma 4.2.5 and Corollary 4.2.6.

Lemma 4.3.8. *Let $R = k[x_1, \ldots, x_n]$ and let $I = \langle x_1^{e_1}, x_2^{e_2}, \ldots, x_n^{e_n} \rangle$ where $e_i \in \mathbb{N} - \{0\}$ for $1 \leq i \leq n$. Then $\mathcal{H}_{\mathbf{R/I}}$ has length $\left(\sum_{j=1}^{n} e_j\right) - n + 1$.*

Proof. It will be sufficient to show that $\mathcal{H}\Big((R/I, \big(\sum_{j=1}^{n} e_j\big) - n\Big) > 0$ and that $\mathcal{H}\Big(R/I, \big(\sum_{j=1}^{n} e_j\big) - n + 1\Big) = 0$.

Let $\mathbf{x}^\alpha = x_1^{e_1-1}x_2^{e_2-1}\cdots x_n^{e_n-1}$. Note that $|\alpha| = \big(\sum_{j=1}^{n} e_j\big) - n$. By Lemma 4.2.5, $\mathbf{x}^\alpha \in R/I$. Hence $\mathcal{H}\Big(R/I, \big(\sum_{j=1}^{n} e_j\big) - n\Big) \geq 1 > 0$.

Now let $\mathbf{x}^\beta$ be a monomial of total degree $\big(\sum_{j=1}^{n} e_j\big) - n + 1$. By the contrapositive of Corollary 4.2.6, then $\mathbf{x}^\beta \notin R/I$. Hence there exists no monomial of total degree $\big(\sum_{j=1}^{n} e_j\big) - n + 1$ in R/I and so $\mathcal{H}\Big(R/I, \big(\sum_{j=1}^{n} e_j\big) - n + 1\Big) = 0$. □

Example 4.3.9. Let $R = k[x, y, z]$ and $I = \langle x^2, y^3, z^4\rangle$. In Example 4.3.7, we saw that $\mathcal{H}_{\mathbf{R/I}}$ had length 7. According to Lemma 4.3.8, $\mathcal{H}_{\mathbf{R/I}}$ has length $(2+3+4) - 3 + 1 = 7$ which agrees with our previous answer. ♢

Note that the sequence in Example 4.3.7 is symmetric. In fact, whenever $I = \langle x_1^{e_1}, \ldots, x_n^{e_n}\rangle$, then the Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ is symmetric in the non-zero entries.

Theorem 4.3.10. *Let $R = k[x_1, \ldots, x_n]$ and let $I = \langle x_1^{e_1}, x_2^{e_2}, \ldots, x_n^{e_n}\rangle$ where $e_m \in \mathbb{N} - \{0\}$ for $1 \leq m \leq n$. Then $\mathcal{H}(R/I, i) = \mathcal{H}\Big(R/I, \big(\sum_{j=1}^{n} e_j\big) - n - i\Big)$ for $0 \leq i \leq \lfloor \frac{(\sum_{j=1}^{n} e_j) - n}{2} \rfloor$.*

Proof. Let $\mathbf{x}^\gamma = x_1^{e_1-1}x_2^{e_2-1}\cdots x_n^{e_n-1}$ and so $|\gamma| = \big(\sum_{j=1}^{n} e_j\big) - n$. By Lemma 4.2.5, then $\mathbf{x}^\gamma \in R/I$ and by Corollary 4.2.6, then $\mathbf{x}^\gamma$ has the maximum total degree that any monomial in R/I can have. For a fixed $i \in \{0, \ldots, \lfloor \frac{(\sum_{j=1}^{n} e_j) - n}{2} \rfloor\}$, define the sets

$$A_i = \{\mathbf{x}^\alpha \mid \mathbf{x}^\alpha \in R/I \text{ and } |\alpha| = i\},$$

and

$$B_i = \{\mathbf{x}^\beta \mid \mathbf{x}^\beta \in R/I \text{ and } |\beta| = \big(\sum_{j=1}^{n} e_j\big) - n - i\}$$

Define the following map $\phi\colon A_i \longrightarrow B_i$ by

$$\phi(\mathbf{x}^\alpha) = \frac{\mathbf{x}^\gamma}{\mathbf{x}^\alpha}.$$

We want to show that ϕ is a bijection so we can prove that $|A_i| = |B_i|$. First, we must show that ϕ is well-defined. Thus we have to show that $\frac{\mathbf{x}^\gamma}{\mathbf{x}^\alpha} \in B_i$ for all $\mathbf{x}^\alpha \in A_i$. Note that since $\mathbf{x}^\gamma, \mathbf{x}^\alpha \in R/I$ and $\mathbf{x}^\gamma$ has the maximum total degree possible in R/I, then $e_m - 1 \geq \alpha_m$ for $1 \leq m \leq n$. Therefore, $\frac{\mathbf{x}^\gamma}{\mathbf{x}^\alpha}$ is actually equal to the monomial $\mathbf{x}^{\gamma-\alpha}$. Thus $\mathbf{x}^{\gamma-\alpha}\mathbf{x}^\alpha = \mathbf{x}^\gamma$ which implies that $\mathbf{x}^{\gamma-\alpha}$ divides $\mathbf{x}^\gamma$. By Corollary 4.2.3, then $\mathbf{x}^{\gamma-\alpha} \in R/I$. Also, we have that the total degree of $\mathbf{x}^{\gamma-\alpha}$ is

$|\gamma| - |\alpha| = \left(\sum_{j=1}^{n} e_j\right) - n - i$ and so we conclude that $\mathbf{x}^{\gamma-\alpha} \in B_i$ for all $\mathbf{x}^{\alpha} \in A_i$. Thus ϕ is well-defined.

Now we will show that ϕ is injective. Suppose $\mathbf{x}^{\alpha}, \mathbf{x}^{\delta} \in A_i$ with $\phi(\mathbf{x}^{\alpha}) = \phi(\mathbf{x}^{\delta})$. Hence $\frac{\mathbf{x}^{\beta}}{\mathbf{x}^{\gamma}} = \frac{\mathbf{x}^{\beta}}{\mathbf{x}^{\delta}}$ implies that $\mathbf{x}^{\gamma}\mathbf{x}^{\alpha} = \mathbf{x}^{\gamma}\mathbf{x}^{\delta}$ and so $\mathbf{x}^{\alpha} = \mathbf{x}^{\delta}$. Thus ϕ is injective.

To complete the proof, we only need to show that ϕ is surjective. We know that since $\mathbf{x}^{\gamma}$ has the largest possible degree in R/I, then for all $\mathbf{x}^{\delta} \in R/I$, we have $\mathbf{x}^{\delta}|\mathbf{x}^{\gamma}$ and $\mathbf{x}^{\gamma-\delta} \in R/I$. Thus if $x^{\beta} \in B_i$, then $\mathbf{x}^{\gamma-\beta} \in R/I$ and $\mathbf{x}^{\gamma-\beta}$ has total degree

$$|\gamma - \beta| = |\gamma| - |\beta| = \Big(\big(\sum_{j=1}^{n} e_j\big) - n\Big) - \Big(\big(\sum_{j=1}^{n} e_j\big) - n - i\Big).$$

If we let $\alpha = \gamma - \beta$, then it follows that $\mathbf{x}^{\alpha} \in A_i$. Thus

$$\phi(\mathbf{x}^{\alpha}) = \frac{\mathbf{x}^{\gamma}}{\mathbf{x}^{\alpha}} = \frac{\mathbf{x}^{\gamma}}{\mathbf{x}^{\gamma-\beta}} = \mathbf{x}^{\beta}.$$

We conclude that ϕ is surjective.

Thus ϕ is bijective, and so $|A_i| = |B_i|$. This means that every monomial of total degree i in R/I corresponds to exactly one monomial of total degree $\left(\sum_{j=1}^{n} e_j\right) - n - i$ in R/I. Therefore, $\mathcal{H}(R/I, i) = \mathcal{H}\Big(R/I, \left(\sum_{j=1}^{n} e_j\right) - n - i\Big)$ for $0 \le i \le \lfloor \frac{\left(\sum_{j=1}^{n} e_j\right) - n}{2} \rfloor$. ☐

Example 4.3.11. As in Example 4.2.7, let $R = k[x, y, z]$ and $I = \langle x^2, y^3, z^4 \rangle$. We saw in Example 4.3.7 that $\mathcal{H}_{\mathbf{R/I}} = (1, 3, 5, 6, 5, 3, 1)$. Theorem 4.3.10 tells us that $\mathcal{H}(R/I, i) = \mathcal{H}(R/I, (2 + 3 + 4) - 3 - i) = \mathcal{H}(R/I, 6 - i)$ for $0 \le i \le \lfloor \frac{(2+3+4)-3}{2} \rfloor = 3$. Thus

$$\mathcal{H}(R/I, 0) = \mathcal{H}(R/I, 6),$$
$$\mathcal{H}(R/I, 1) = \mathcal{H}(R/I, 5),$$
$$\mathcal{H}(R/I, 2) = \mathcal{H}(R/I, 4),$$

and

$$\mathcal{H}(R/I, 3) = \mathcal{H}(R/I, 3).$$

Note that this agrees with the sequence above. ♢

Example 4.3.12. Now let $R = k[x, y, z]$ and $I = \langle x^2, y^3, z^3 \rangle$. We can figure out which monomials are in R/I from the set given in Example 4.2.7 by removing all monomials in $R/\langle x^2, y^3, z^4 \rangle$ that include z^3. Hence the monomials in R/I are

$$\{1, x, y, z, y^2, z^2, xy, xz, yz, xy^2, xz^2, y^2z, yz^2, xyz, y^2z^2, xy^2, xyz^2, xy^2z^2\}.$$

It follows that $\mathcal{H}_{\mathbf{R/I}} = (1, 3, 5, 5, 3, 1)$. Now by using Theorem 4.3.10, we have $\mathcal{H}(R/I, i) = \mathcal{H}(R/I, (2 + 3 + 3) - 3 - i) = \mathcal{H}(R/I, 5 - i)$ for $0 \le i \le \lfloor \frac{(2+3+3)-3}{2} \rfloor = 2$. Thus

$$\mathcal{H}(R/I, 0) = \mathcal{H}(R/I, 5),$$

$$\mathcal{H}(R/I,1) = \mathcal{H}(R/I,4),$$

and

$$\mathcal{H}(R/I,2) = \mathcal{H}(R/I,3),$$

as we saw in the above sequence. ◊

Theorem 4.3.10 shows us that in order to find $\mathcal{H}_{\mathbf{R/I}}$ for ideals of the type given in Theorem 4.3.10, we only need to compute the number of monomials in R/I of total degree less than or equal to $\lfloor \frac{\left(\sum_{j=1}^{n} e_j\right)-n}{2} \rfloor$.

Theorem 4.3.13. *Let $R = k[x_1,\ldots,x_n]$ and suppose I is a monomial ideal. Then $\mathcal{H}_{\mathbf{R/I}}$ is finite if and only if for each $0 \le i \le n$, there exists $e_i \in \mathbb{N}$ with $x_i^{e_i} \in I$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{H}_{\mathbf{R/I}}$ is finite. By definition, this means that $\mathcal{H}(R/I,m) = 0$ for some $m \in \mathbb{N}$ and so if $\mathbf{x}^\beta$ has total degree m, then $\mathbf{x}^\beta \in I$. In particular, $x_1^m, x_2^m, \ldots, x_n^m \in I$.

($\Leftarrow$) Now suppose that $x_1^{e_1}, x_2^{e_2}, \ldots, x_n^{e_n} \in I$ for some $e_1, e_2, \ldots, e_n \in \mathbb{N}$. By Corollary 4.2.6, if $\mathbf{x}^\beta \in R/I$, then $|\beta| \le \left(\sum_{j=1}^n e_j\right) - n$. By the contrapositive of the same corollary, if $\mathbf{x}^\beta$ is any monomial of total degree strictly greater than $\left(\sum_{j=1}^n e_j\right) - n$, then $\mathbf{x}^\beta \in I$. Then $\mathcal{H}\Big(R/I, \left(\sum_{j=1}^n e_j\right) - n + 1\Big) = 0$ and so $\mathcal{H}_{\mathbf{R/I}}$ is finite. □

It is worth noting that in Theorem 4.3.13, $\mathcal{H}(R/I,m)$ may equal 0 for some $m < \left(\sum_{j=1}^n e_j\right) - n + 1$, but at the very least, $\mathcal{H}\Big(R/I, \left(\sum_{j=1}^n e_j\right) - n + 1\Big) = 0$.

We have seen illustrations of Theorem 4.3.13 throughout this project. The idea behind the proof is that if I does not contain a monomial of the form $x_i^{e_i}$ for some $1 \le i \le n$ and $e_i \in \mathbb{N}$, then $x_i^r \in R/I$ for all $r \in \mathbb{N}$. Hence $\mathcal{H}(R/I,p) \ge 1$ for all $p \in \mathbb{N}$ which shows that the sequence $\mathcal{H}_{\mathbf{R/I}}$ is infinite.

Example 4.3.14. Let $R = k[x,y]$ and $I = \langle x^4, x^3y, xy^2, y^3\rangle$. Since, $x^4, y^3 \in I$, Theorem 4.3.13 tells us that $\mathcal{H}_{\mathbf{R/I}}$ should be finite. It is not difficult to show that $\mathcal{H}_{\mathbf{R/I}} = (1,2,3,2)$.

Now let $I' = \langle x^4, x^3y, xy^2\rangle$. Theorem 4.3.13 now tells us that since $y^i \notin I'$ for all $i \ge 0$, then $\mathcal{H}_{\mathbf{R/I'}}$ should be infinite. We can also easily show that $\mathcal{H}_{\mathbf{R/I'}} = (1,2,3,3,1,1,1,\ldots)$, which is indeed infinite. ◊

4.4 The 1-dimensional Case

In this section, we will only consider the polynomial ring $R = k[x]$. We will determine all possible sequences $\mathcal{H}_{\mathbf{R/I}}$ of monomial ideals in R. Also, given any sequence for $\mathcal{H}_{\mathbf{R/I}}$, we will be able to construct a monomial ideal with the desired sequence.

Theorem 4.4.1. *Suppose that $R = k[x]$.*

1. *If I is a monomial ideal in R, then* $\mathcal{H}_{\mathbf{R/I}} = \underbrace{(1,1,1,\ldots,1)}_{n\text{-times}}$ *for some* $n \in \mathbb{N}$.
2. *If* $m \in \mathbb{N}$*, then there is some monomial ideal* I' *in* R *such that* $\mathcal{H}_{\mathbf{R/I}} = \underbrace{(1,1,1,\ldots,1)}_{m\text{-times}}$.

   In particular, if $I' = \langle x^m \rangle$ *then we obtain the desired sequence for* $\mathcal{H}_{\mathbf{R/I}}$.

Proof. Recall that by Proposition 2.2.6, $\mathcal{H}(R,i) = \binom{1+i-1}{i} = \binom{i}{i} = 1$ for all $i \geq 0$ and so $\mathcal{H}_{\mathbf{R}} = (1,1,1,1,1,\ldots)$. This makes sense as we saw in Example 2.2.2 that $R_i = \operatorname{span}\{x^i\}$ for all $i \geq 0$, and so $\dim_k(R_i) = 1$. Thus either $\mathcal{H}(R/I,i) = 0$ or $\mathcal{H}(R/I,i) = 1$ for all $i \in \mathbb{N}$.

If $\mathcal{H}(R/I,j) = 0$ for the least such $j \in \mathbb{N}$, then Theorem 4.3.1 states that $\mathcal{H}(R/I,m) = 0$ for all $m \geq j$. Since j is the least such positive integer with the desired property, then $\mathcal{H}(R/I,j-1) \neq 0$ and so $\mathcal{H}(R/I,j-1) = 1$. By the repeated use of Corollary 4.3.2, $\mathcal{H}(R/I,n) = 1$ for $0 \leq n < j$.

Thus for any monomial ideal I in R, then $\mathcal{H}_{\mathbf{R/I}}$ is one of the following sequences:

$$(0)$$
$$(1)$$
$$(1,1)$$
$$(1,1,1)$$
$$(1,1,1,1)$$
$$(1,1,1,1,1)$$
$$\vdots$$
$$\underbrace{(1,1,1,\ldots,1)}_{n\text{-times}}.$$

Now we must find a monomial ideal in R that gives us each of these sequences listed above. If we want $\mathcal{H}_{\mathbf{R/I}} = (0)$, then it must be that case that $\mathcal{H}(R/I,0) = 0$ and so $\mathcal{H}(I,0) = 1$. Thus I must contain one monomial of total degree 0 and so if $I = \langle x^0 \rangle = \langle 1 \rangle = k[x]$, then $\mathcal{H}_{\mathbf{R/I}} = (0)$. Similarly, if we want $\mathcal{H}_{\mathbf{R/I}} = \underbrace{(1,1,1,\ldots,1)}_{m\text{-times}}$, then $\mathcal{H}(R/I,m) = 0$ and so $\mathcal{H}(I,m) = 1$. Thus I must contain one monomial of total degree m and so if $I = \langle x^m \rangle$, then $\mathcal{H}_{\mathbf{R/I}} = \underbrace{(1,1,1,\ldots,1)}_{m\text{-times}}$. Note that both of these cases follow from Lemma 4.3.8. Therefore, each sequence $\underbrace{(1,1,1,\ldots,1)}_{m\text{-times}}$ can be obtained by computing $\mathcal{H}_{\mathbf{R/I}}$ where I is the monomial ideal $\langle x^m \rangle$.

Note that there is one other case that we did not address above. This is the case where $\mathcal{H}_{\mathbf{R/I}} = (1,1,1,1,\ldots)$. However, this implies that $R/I \cong R$ and so $I = \{0\}$. Since 0 is not a monomial, I is not a monomial ideal and so we dismiss this case. ☐

Example 4.4.2. Let $I = \langle x^3 \rangle$. Then $I_0 = I_1 = I_2 = \{0\}$ and $I_i = \text{span}\{x^i\}$ for all $i \geq 3$. Thus $\mathcal{H}_{\mathbf{I}} = (0, 0, 0, 1, 1, 1, 1, \ldots)$ and so $\mathcal{H}_{\mathbf{R/I}} = (1, 1, 1)$. This agrees with Theorem 4.4.1. ♢

Example 4.4.3. Let $I = \langle x^3, x^7, x^{10} \rangle$. It is not difficult to see that $\mathcal{H}_{\mathbf{I}} = (0, 0, 0, 1, 1, 1, 1, \ldots)$ and $\mathcal{H}_{\mathbf{R/I}} = (1, 1, 1)$ as we found in the last example. The reason for this is that $k[x]$ is a principal ideal domain which means that every ideal in $k[x]$ can be generated by a single element. Here we see that $x^7 = x^4(x^3)$ and $x^{10} = x^7(x^3)$ so the generators x^7 and x^{10} are superfluous. Thus $I = \langle x^3, x^7, x^{10} \rangle = \langle x^3 \rangle$. In fact, it is true in general that if $I = \langle x^{a_1}, x^{a_2}, \ldots, x^{a_n} \rangle$ with $a_1 \leq a_2 \leq \cdots \leq a_n$, then $I = \langle x^{a_1} \rangle$ since each x^{a_i} is simply a monomial multiple of x^{a_1} for $1 \leq i \leq n$. ♢

It is interesting to note that while $k[x]$ is a principal ideal domain, $k[x, y]$ is not. For example, for $I = \langle x^2, y^2 \rangle$, there is no way that we can generate I with a single element. If we were to generate I with a monomial in x alone, there would be no way we would ever be able to generate the term y^2 in I. Similarly, if we were to generate I with a monomial in y alone, we could not generate the term x^2 in I. If we were to generate I with a monomial that included both x and y, we would not be unable to generate either x^2 or y^2 in I. In fact, $k[x_1, \ldots, x_n]$ fails to be a principal ideal domain for each $n > 1$.

This concludes our discussion of Hilbert sequences of monomial ideals in $k[x]$.

4.5 The 2-dimensional Case

In this section, we will be working only in $R = k[x, y]$. Our goal is to identify all possible Hilbert sequences for R/I where I is a monomial ideal in R. We begin by first investigating some properties of the Hilbert function in $k[x, y]$. Then, we will discuss all possible finite Hilbert sequences $\mathcal{H}_{\mathbf{R/I}}$. Finally, we investigate the possible infinite Hilbert sequences for R/I and will construct monomial ideals which have these sequences.

Throughout this project, we have been computing many examples in $k[x, y]$. In all of these examples of Hilbert sequences $\mathcal{H}_{\mathbf{R/I}}$, the sequences increase for a while, then either decrease until they eventually become 0, decrease until they stay at one value forever, or stay at the largest value of the increase. In addition, once the sequence begins decreasing or staying at a fixed value, it never increases again. We will show that this is true for all monomial ideals in $k[x, y]$.

Recall that since $I \subseteq R = k[x, y]$, then $\mathcal{H}(R, i) = i + 1$ for all $i \in \mathbb{N}$. Thus $\mathcal{H}(I, i) \leq i+1$ and $\mathcal{H}(R/I, i) \leq i+1$ because $\mathcal{H}(I, i) + \mathcal{H}(R/I, i) = \mathcal{H}(R, i) = i+1$.

We begin by noting that whenever $\mathcal{H}(R/I, i) < i+1$, we have that $\mathcal{H}(R/I, i+r) < i+1$ for all $r \in \mathbb{N}$. This shows us that once the entries of $\mathcal{H}_{\mathbf{R/I}}$ are less than the maximum number possible, the remaining entries in the sequence will never be greater than the value at which the sequence begins decreasing. We will now prove this fact.

Theorem 4.5.1. *Let $R = k[x,y]$ and let I be any monomial ideal in R. If $\mathcal{H}(R/I,i) \leq i$ for some $i \in \mathbb{N}$, then $\mathcal{H}(R/I,i+r) \leq i$ for all $r \in \mathbb{N}$.*

Proof. Suppose $\mathcal{H}(R/I,i) \leq i$ for some $i \in \mathbb{N}$. Then $\mathcal{H}(R,i) - \mathcal{H}(I,i) \leq i$ since $\mathcal{H}(R/I,j) = \mathcal{H}(R,j) - \mathcal{H}(I,j)$ for all $j \geq 0$. Since $\mathcal{H}(R,i) = i+1$, we have

$$(i+1) - \mathcal{H}(I,i) \leq i,$$

and so

$$-\mathcal{H}(I,i) \leq i - (i+1) = -1,$$

and finally

$$\mathcal{H}(I,i) \geq 1.$$

This means that there exists at least one monomial of total degree i in I. Let $x^a y^b$ be one such monomial such that $a+b=i$ where $a,b \in \mathbb{N}$. The monomials $(x^a y^b)(x^c y^d)$ will be in I for all $c,d \in \mathbb{N}$. Then the monomials in I of total degree $i+r$ will include the monomials $(x^a y^b)(x^c y^d)$ where $c+d=r$ and so $\mathcal{H}(I,i+r)$ is at least as large as the total number of monomials of this type. Since $x^a y^b$ is fixed, then we only need to figure out how many such $x^c y^d$ there are for a given r. Note that this is simply the number of monomials of total degree r and by Theorem 2.2.6, this number is $\binom{r+2-1}{r} = \binom{r+1}{r} = r+1$. Therefore, $\mathcal{H}(I,i+r) \geq r+1$ and so

$$\mathcal{H}(R,i+r) - \mathcal{H}(R/I,i+r) \geq r+1.$$

Since $\mathcal{H}(R,i+r) = (i+r)+1$, then

$$(i+r+1) - \mathcal{H}(R/I,i+r) \geq r+1$$

which implies that

$$-\mathcal{H}(R/I,i+r) \geq r+1-(i+r+1) = -i$$

and so finally $\mathcal{H}(R/I,i+r) \leq i$ for all $r \in \mathbb{N}$. □

Example 4.5.2. The sequence $(1,2,3,4,4,4,4,\ldots)$ does not violate Theorem 4.5.1 and so it could possibly be a Hilbert sequence. In fact, it follows from Theorem 4.3.3 that this is the sequence for $\mathcal{H}_{\mathbf{R/I}}$ where I is generated by any single monomial of total degree 4. However, this theorem also shows us that the sequence $(1,2,3,4,4,5,5,5,\ldots)$ is not a Hilbert sequence. Note that the sequence $(1,2,3,4,4,3,3,2,4,4,\ldots)$ does not violate the criteria given in the theorem above. We will show in Theorem 4.5.5 that this is not a Hilbert sequence since the entries begin decreasing but then increase again. Therefore, this theorem can only tell us if a sequence is definitely not a Hilbert sequence. ◊

To prove that once the entries in a Hilbert sequence starts decreasing or plateauing, then they never increase again, we must first make some observations about $\mathcal{H}(I,i)$ for $i \in \mathbb{N}$. We see from our previous examples that once $\mathcal{H}(I,i) > 0$, then

$$\mathcal{H}(I,i) < \mathcal{H}(I,i+1) < \mathcal{H}(I,i+2) < \mathcal{H}(I,i+3) < \cdots .$$

This means that once the entries in $\mathcal{H}_{\mathbf{I}}$ are non-zero, the remaining entries are strictly increasing. The reason why this result is true relies on the fact that for every monomial $m \in I$ of total degree s, we can multiply that m by either x or y to get two distinct monomials in I of total degree $s+1$.

Theorem 4.5.3. *Let $R = k[x,y]$ and suppose that I is any monomial ideal in R. If $\mathcal{H}(I,i) = n$ for some $i \geq 0$ and $n \in \mathbb{N} - \{0\}$, then $\mathcal{H}(I,i+r) \geq n+r$ for all $r \in \mathbb{N}$.*

Proof. We will prove this by induction on n. Suppose that $n = 1$. This means that there exists exactly one monomial $m \in I$ of total degree i. Then $m = x^a y^b$ where $a+b = i$. We know that all monomials of the form $(x^a y^b)(x^c y^d)$ will be in I where $c, d \geq 0$. Thus, if we want to find the monomials of total degree $i+r$ in I, we know that there will be at least the monomials $(x^a y^b)(x^c y^d)$ for all c and d such that $c+d = r$. Now to find $\mathcal{H}(I,i+r)$, we simply need to find how many such monomials there are for a given r. Since $x^a y^b$ is fixed, we see that the number of monomials of the form $(x^a y^b)(x^c y^d)$ with $c+d = r$ is simply the number of monomials in x and y with total degree r. By Theorem 2.2.6, there are $\binom{r+2-1}{r} = \binom{r+1}{r} = r+1$ such monomials. It is possible that there are other monomials in I that are added by generators with total degree larger than i, and this could possibly make $\mathcal{H}(I,i+r)$ larger. Therefore, $\mathcal{H}(I,i+r) \geq 1+r$ and so the result holds.

Now assume that the result is true for $n-1$. That is, if $\mathcal{H}(J,i) = n-1$ for any monomial ideal J, then $\mathcal{H}(J,i+r) \geq n-1+r$. Let I be a monomial ideal such that $\mathcal{H}(I,i) = n$. Let $M = \{m_1, m_2, \cdots, m_n\}$ be the set of distinct monomials of total degree i in I, and by hypothesis, we see that $|M| = n$. Notice that each $m_j \in M$ takes the form $m_j = x^{a_j} y^{i-a_j}$ for some unique $a_j \in \mathbb{N}$. Let a_p be the maximum such a_j and so $m_p = x^{a_p} y^{i-a_p}$ is the monomial in I of total degree i with the largest possible exponent for x.

Now let $I' = \langle M - \{m_p\} \rangle$. It follows that every monomial in I' is also in I so $I' \subset I$. Since both I' and I are standard graded k-algebras, then $I'_t \subseteq I_t$ for all $t \in \mathbb{N}$. In particular, $I'_{i+r} \subseteq I_{i+r}$ and so by Theorem 2.1.4, then $\dim_k(I'_{i+r}) \leq \dim_k(I_{i+r})$. This means that $\mathcal{H}(I', i+r) \leq \mathcal{H}(I, i+r)$.

We now consider monomials of total degree i in I'. Because of the way that we chose to construct I', then there are $n-1$ such monomials and so $\mathcal{H}(I',i) = n-1$. We can utilize our hypothesis and we find that $\mathcal{H}(I', i+r) \geq n-1+r$. Also, note that all monomials of total degree i in I' are also in I and they take the form $x^{a_p-e} y^{i-a_p+e}$ where $1 \leq e \leq a_p$. Otherwise, $x^{a_p} y^{i-a_p}$ would not have the largest possible exponent for x in I. Now, the monomial of total degree $i+r$ with the largest possible exponent for x in I' is $x^r(x^{a_p-e} y^{i-a_p+e})$ where e is the least such

$e \in \{1, \ldots, a_p\}$ such that $x^{a_p - e} y^{i - a_p + e} \in I'$. However, the monomial of total degree $i + r$ with the largest possible exponent for x in I is $x^r(x^{a_p} y^{i-a_p})$. Note that this monomial cannot be in I' and so there is at least one monomial in I that is not in I'. Therefore, $\mathcal{H}(I, i + r) \geq \mathcal{H}(I', i + r) + 1 \geq (n - 1 + r) + 1 = n + r$ for all $r \in \mathbb{N}$. □

Example 4.5.4. Suppose that x^5 and $x^4 y$ are the only monomials of total degree 5 in I. Then clearly $\mathcal{H}(I, 5) = 2$. It is easy to see that the monomials $x(x^5) = x^6$, $y(x^5) = x(x^4 y) = x^5 y$, and $y(x^4 y) = x^4 y^2$ are all monomials of total degree 6 in I. However, it is possible that there are monomials in I other than the ones listed above. We know at the very least that $\mathcal{H}(I, 6) \geq 3 = 2 + 1$. Similarly, we can show that the monomials in I of total degree $5 + r$ with $r \in \mathbb{N}$ contain the set

$$\{x^{5+r}, x^{4+r}y, x^{3+r}y^2, x^{2+r}y^3, x^{1+r}y^4, x^r y^5, x^{r-1}y^6, \ldots, x^5 y^r, x^4 y^{r+1}\},$$

and there are $2 + r$ such monomials. Thus $\mathcal{H}(I, 5 + r) \geq 2 + r$. This agrees with Theorem 4.5.3 which tells us that since $\mathcal{H}(I, 5) \geq 2$, then $\mathcal{H}(I, 5 + r) \geq 2 + r$ for all $r \in \mathbb{N}$. ♢

Note that Theorem 4.5.3 also shows us that $\mathcal{H}_{\mathbf{I}}$ is an infinite sequence unless $\mathcal{H}_{\mathbf{I}} = (0)$ in which case $I = \{0\}$. We are now ready to show that once the entries in $\mathcal{H}_{\mathbf{R/I}}$ begin decreasing, they never increase again.

Theorem 4.5.5. *Let $R = k[x, y]$ and let I be any monomial ideal in R. If $\mathcal{H}(R/I, i) = m < i + 1$, then $\mathcal{H}_{\mathbf{R/I}}$ for all $r \in \mathbb{N}$.*

Proof. Let $\mathcal{H}(R/I, i) = m$. Since $\mathcal{H}(R, i) - \mathcal{H}(I, i) = \mathcal{H}(R/I, i)$ and $\mathcal{H}(R, i) = i + 1$, then letting $\mathcal{H}(I, i) = n$, we have $n + m = i + 1$. Since $m < i + 1$, it follows that $n > 0$. By Theorem 4.5.3, this implies that $\mathcal{H}(I, i + r) \geq n + r$ for all $r \in \mathbb{N}$. Thus

$$\mathcal{H}(R, i + r) - \mathcal{H}(R/I, i + r) \geq n + r,$$

and since $\mathcal{H}(R, i + r) = i + r + 1$,

$$(i + r + 1) - \mathcal{H}(R/I, i + r) \geq n + r,$$

which implies that

$$-\mathcal{H}(R/I, i + r) \geq (n + r) - (i + r + 1) = n - i - 1,$$

and so finally we find that

$$\mathcal{H}(R/I, i + r) \leq -n + i + 1 = -n + (n + m) = m$$

for all $r \in \mathbb{N}$, as was desired. □

Corollary 4.5.6. *Let $R = k[x,y]$ and suppose that I is a monomial ideal in R. If $\mathcal{H}(R/I, i) < i+1$ for some $i \in \mathbb{N}$, then the sequence $\mathcal{H}_{\mathbf{R/I}}$ never increases beyond the $(i+1)$-th entry.*

Proof. This corollary is simply the repeated application of Theorem 4.5.5. □

Corollary 4.5.7. *Any sequence that cannot be written as $(1, 2, 3, \ldots, n, n_1, n_2, n_3, n_4, \ldots)$ with $n, n_1, n_2, n_3, n_4, \ldots \in \mathbb{N}$ and $n \geq n_1 \geq n_2 \geq n_3 \geq n_4 \geq \cdots$ is not a Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ for any monomial ideal I in $R = k[x,y]$.*

Proof. This corollary relies on a single observation. Note that Theorem 4.5.5 tells us that once we have any case in which

$$\mathcal{H}(R/I, i) = n_1 < i+1,$$

then

$$\mathcal{H}(R/I, j) \leq n_1$$

for any $j \geq i$. However, now if $\mathcal{H}(R/I, j) = n_2$, we see that

$$\mathcal{H}(R/I, j) = n_2 \leq n_1 < i+1 \leq j+1.$$

Using Theorem 4.5.5 again, we have

$$\mathcal{H}(R/I, m) \leq n_2$$

for any $m \geq j$. Repeating this idea, we see that the corollary follows directly from Theorem 4.5.5. □

Example 4.5.8. We now see that the sequence $(1, 2, 3, 4, 4, 3, 3, 2, 4, 4, \ldots)$ given in Example 4.5.2 is a not a Hilbert sequence. If the sequence were a Hilbert sequence, then since $\mathcal{H}(R/I, 5) = 3$ and $3 < 5+1 = 6$, Theorem 4.5.5 states that $\mathcal{H}(I, 5+r) \leq 3$ for all $r \in \mathbb{N}$. However, in the sequence we have $\mathcal{H}(I, 8) = 4$, a contradiction. We conclude that $(1, 2, 3, 4, 4, 3, 3, 2, 4, 4, \ldots)$ is not a Hilbert sequence. Similarly, we see that for $A = (1, 2, 3, 3, 3, 4, 5, 6, 7, \ldots)$, $B = (1, 2, 3, 4, 4, 5, 4, 3, 2, 1)$, and $C = (1, 2, 3, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, \ldots)$ there does not exist a monomial ideal I in $R = k[x,y]$ such that $\mathcal{H}_{\mathbf{R/I}}$ is A, B, or C. Thus A, B, and C are not Hilbert sequences. ◊

The most important use of Corollary 4.5.7 is that it tells us that any finite sequence of the form

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, \ldots, \underbrace{4, \ldots, 4}_{k_4 \text{ times}}, \underbrace{3, \ldots, 3}_{k_3 \text{ times}}, \underbrace{2, \ldots, 2}_{k_2 \text{ times}}, \underbrace{1, \ldots, 1}_{k_1 \text{ times}})$$

where $k_n > 0$ and $k_i \geq 0$ for $1 \leq i \leq n-1$, is consistent with the criteria given in Theorem 4.5.1 and Theorem 4.5.5. Thus, we conclude that the sequence is possibly a Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ for some monomial ideal I, if we can find the proper I.

Similarly, any infinite sequence of the form

$$(1, 2, 3, \ldots, n, n, n, \ldots)$$

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, n-1, n-1, n-1, \ldots)$$

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, n-2, n-2, n-2, \ldots)$$

$$\vdots$$

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, \ldots, \underbrace{4, \ldots, 4}_{k_4 \text{ times}}, \underbrace{3, \ldots, 3}_{k_3 \text{ times}}, \underbrace{2, \ldots, 2}_{k_2 \text{ times}}, 1, 1, 1, \ldots)$$

with $k_n > 0$ and $k_i \geq 0$ for $2 \leq i \leq n-1$, is possibly a Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$, if such an ideal I exists.

We now investigate finite sequences of the form

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, \ldots, \underbrace{4, \ldots, 4}_{k_4 \text{ times}}, \underbrace{3, \ldots, 3}_{k_3 \text{ times}}, \underbrace{2, \ldots, 2}_{k_2 \text{ times}}, \underbrace{1, \ldots, 1}_{k_1 \text{ times}})$$

where $k_n > 0$ and $k_i \geq 0$ for $1 \leq i \leq n-1$. Note that after the first n entries, each entry that follows is less than or equal to the entry immediately proceeding it. We will show that given a fixed length $i = (n-1) + \sum_{j=1}^{n} k_j$, then there are 2^{i-1} different sequences of length i of the form

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, \ldots, \underbrace{4, \ldots, 4}_{k_4 \text{ times}}, \underbrace{3, \ldots, 3}_{k_3 \text{ times}}, \underbrace{2, \ldots, 2}_{k_2 \text{ times}}, \underbrace{1, \ldots, 1}_{k_1 \text{ times}})$$

where $k_1, k_2, \ldots, k_n \in \mathbb{N}$.

Theorem 4.5.9. *Let S_i be the set of all sequences of the form $(1, 2, 3, \ldots, n, n_1, n_2, \ldots, n_{i-n})$ where $n \geq n_1 \geq n_2 \geq \cdots \geq n_{i-n} > 0$ and $1 \leq n \leq i$. Then $|S_i| = 2^{i-1}$.*

Proof. Note that the sequence $(1, 2, 3, \ldots, n, n_1, n_2, \ldots, n_{i-n})$ has length i and for a given n, the first n entries of this sequence are fixed. However, to construct the last $i-n$ entries of this sequence, let $A = \{a_1, a_2, \ldots, a_{i-n}\}$ be a multiset where $1 \leq a_j \leq n$ for $1 \leq j \leq i-n$. We can reorder the elements of A in decreasing order and relabel these $n_1, n_2, \ldots, n_{i-n}$ where $n \geq n_1 \geq n_2 \geq \cdots \geq n_{i-n}$. Thus for a given n, the number of sequences of length i is simply the number of possible multisets A. In combinatorial terms, this is the number of distributions of $i-n$ identical objects to n distinct recipients which is

$$\binom{n+(i-n)-1}{i-n} = \binom{i-1}{i-n} = \binom{i-1}{(i-1)-(i-n)} = \binom{i-1}{n-1}$$

(see [2, Theorem 2.3] for example).

Also, since i is fixed and $1 \leq n \leq i$, we have

$$|S_i| = \sum_{n=1}^{i} \binom{i-1}{n-1} = \sum_{n=0}^{i-1} \binom{i-1}{n} = \binom{i-1}{0} + \binom{i-1}{1} + \binom{i-1}{2} + \cdots + \binom{i-1}{i-1}.$$

Using results from combinatorics once again, we know that $\binom{i-1}{j}$ is the number of j-element subsets of an $(i-1)$-element set (see [2, Theorem 3.2]). Hence the above sum is the total number of subsets of an $(i-1)$-element set, which equals 2^{i-1} (see [2, Theorem 3.1]). We conclude that $|S_i| = 2^{i-1}$. □

Notice that if a finite sequence cannot be written in the form $(1, 2, 3, \dots, n, n_1, n_2, \dots, n_{i-n})$, then by Theorem 4.5.5, the sequence is not a Hilbert sequence for any monomial ideal I in $k[x, y]$.

Example 4.5.10. We will find all sequences of length 4 that have the form

$$(1, 2, 3, \dots, n, n_1, n_2, \dots, n_{4-n})$$

with $n \geq n_1 \geq n_2 \geq \cdots \geq n_{4-n}$ and $1 \leq n \leq 4$.

If $n = 1$, then all such sequences have the form $(1, n_1, n_2, n_3)$ where $1 \geq n_1 \geq n_2 \geq n_3 > 0$. It must be the case that $n_1 = n_2 = n_3 = 1$ and so the only sequence we find is $(1, 1, 1, 1)$.

If $n = 2$, then all sequences have the form $(1, 2, n_1, n_2)$ where $2 \geq n_1 \geq n_2 > 0$. If $n_1 = 2$, then either $n_2 = 2$ or $n_2 = 1$, and so we obtain the sequences $(1, 2, 2, 2)$ and $(1, 2, 2, 1)$. If $n_1 = 1$, then $n_2 = 1$ and so we get the sequence $(1, 2, 1, 1)$.

If $n = 3$, we obtain all sequences of the form $(1, 2, 3, n_1)$ where $3 \geq n_1 > 0$. Thus $n_1 = 1$, $n_1 = 2$, or $n_1 = 3$ and so we obtain the sequences $(1, 2, 3, 3)$, $(1, 2, 3, 2)$, and $(1, 2, 3, 1)$.

Finally, if $n = 4$, then we have the sequence $(1, 2, 3, 4)$. Therefore,

$$S_4 = \{(1,1,1,1), (1,2,2,2), (1,2,2,1), (1,2,1,1), (1,2,3,3), (1,2,3,2), (1,2,3,1), (1,2,3,4)\}$$

and we see that there are 8 such sequences in this set. This is consistent with Theorem 4.5.9 which states that $|S_4| = 2^{4-1} = 2^3 = 8$.

As it turns out, it is not difficult to show that

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{y}, \mathbf{x^4} \rangle)} = (1, 1, 1, 1),$$

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{xy}, \mathbf{y^4}, \mathbf{x^4} \rangle)} = (1, 2, 2, 2),$$

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{xy}, \mathbf{y^3}, \mathbf{x^4} \rangle)} = (1, 2, 2, 1),$$

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{xy}, \mathbf{y^2}, \mathbf{x^4} \rangle)} = (1, 2, 1, 1),$$

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{x^2y}, \mathbf{xy^3}, \mathbf{y^4}, \mathbf{x^4} \rangle)} = (1, 2, 3, 3),$$

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{x^2y}, \mathbf{xy^2}, \mathbf{y^4}, \mathbf{x^4}\rangle)} = (1,2,3,2),$$
$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{x^2y}, \mathbf{xy^2}, \mathbf{y^3}, \mathbf{x^4}\rangle)} = (1,2,3,1),$$

and

$$\mathcal{H}_{\mathbf{R}/\langle \mathbf{x^3y}, \mathbf{x^2y^2}, \mathbf{xy^3}, \mathbf{y^4}, \mathbf{x^4}\rangle)} = (1,2,3,4).$$

This shows us that all sequences in S_4 are indeed Hilbert sequences of monomial ideals. ♢

Since Theorems 4.5.12 and 4.5.14 are difficult to read through, I will first make a comment that I hope will help the reader to understand some of the details of their proofs.

If we consider the sequence

$$S = (1,2,3,\dots,\underbrace{n,\dots,n}_{k_n \text{ times}},\underbrace{n-1,\dots,n-1}_{k_{n-1} \text{ times}},\dots,\underbrace{4,\dots,4}_{k_4 \text{ times}},\underbrace{3,\dots,3}_{k_3 \text{ times}},\underbrace{2,\dots,2}_{k_2 \text{ times}},\underbrace{1,\dots,1}_{k_1 \text{ times}}),$$

for some fixed $n, k_n \in \mathbb{N} - \{0\}$ and $k_1, k_2, \dots, k_{n-1} \in \mathbb{N}$, we see that the length of S is $n - 1 + \sum_{j=1}^{n} k_j$. Thus we can write

$$S = (e_0, e_1, e_2, \dots e_{n-2+\sum_{j=1}^{n} k_j}).$$

Clearly, $e_i = i + 1$ for $0 \le i \le n - 2$. Also note that the entries in the sequence e_{n-1} through e_{n-2+k_n} are n. Similarly, the entries e_{n-1+k_n} through $e_{n-2+k_n+k_{n-1}}$ are $n-1$ and the entries $e_{n-1+k_n+k_{n-1}}$ through $e_{n-2+k_n+k_{n-1}+k_{n-2}}$ are $n-2$. Repeating this procedure, we find that the entries $e_{n-1+\sum_{j=5}^{n} k_j}$ through $e_{n-2+\sum_{j=4}^{n} k_j}$ are 4, the entries $e_{n-1+\sum_{j=4}^{n} k_j}$ through $e_{n-2+\sum_{j=3}^{n} k_j}$ are 3, the entries $e_{n-1+\sum_{j=3}^{n} k_j}$ through $e_{n-2+\sum_{j=2}^{n} k_j}$ are 2, and finally the entries $e_{n-1+\sum_{j=2}^{n} k_j}$ through $e_{n-2+\sum_{j=1}^{n} k_j}$ are 1. Thinking about S in this manner will help us to locate at which entries our sequence is obtaining a given value. In Theorem 4.5.14, we will actually show that there exists a monomial ideal I in $R = k[x, y]$ such that $\mathcal{H}_{\mathbf{R/I}} = S$ and so it is important to keep in mind which value an entry $e_i = \mathcal{H}(R/I, i)$ is.

Example 4.5.11. To give some clarity to the ideas just described, suppose we are trying to construct a sequence S of the form outlined above where $n = 4$, $k_4 = 3$, $k_3 = 2$, $k_2 = 4$, and $k_1 = 1$. Thus $S = (1,2,3,4,4,4,3,3,2,2,2,2,1)$, but let us now construct the sequence one entry at a time. Using the ideas outlined above, we see that S has length $4-1+(3+2+4+1) = 13$ so we can write S as $(e_0, e_1, \dots, e_{12})$. The statement above also tells us that $e_i = i+1$ for $0 \le i \le 2$, and so $e_0 = 1$, $e_1 = 2$, and $e_2 = 3$. We also have that the entries in the sequence $e_{4-1} = e_3$ through $e_{4-2+3} = e_5$ are 4, the entries $e_{4-1+3} = e_6$ through $e_{4-2+3+2} = e_7$ are 3, the entries $e_{4-1+3+2} = e_8$ through $e_{4-2+3+2+4} = e_{11}$ are 2, and finally the entries $e_{4-1+3+2+4} = e_{12}$ through $e_{4-2+3+2+4+1} = e_{12}$ are 1. Therefore,

$$\begin{aligned} S &= (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}) \\ &= (1,2,3,4,4,4,3,3,2,2,2,2,1) \end{aligned}$$

as we had desired. ♢

We can also use this method to figure out which entries are at a given value in an infinite sequence of the form

$$S' = (1,2,3,\ldots,\underbrace{n,\ldots,n}_{k_n \text{ times}},\underbrace{n-1,\ldots,n-1}_{k_{n-1} \text{ times}},\ldots,\underbrace{n-i+1,\ldots,n-i+1}_{k_{n-i+1} \text{ times}},n-i,n-i,\ldots),$$

for any $n, k_n \in \mathbb{N} - \{0\}$, $0 \leq i \leq n$, and $k_1, k_2, \ldots, k_{n-1} \in \mathbb{N}$. We proceed as we did previously until we come to the entry $e_{n-1+\sum_{j=n-i+1}^{n} k_j}$ which we see is $n-i+1$. We know that every entry that comes after this entry is also $n-i+1$, and so $e_j = n-i+1$ for all $j \geq n-1+\sum_{j=n-i+1}^{n} k_j$.

We are now ready to show that for every such sequence S' with $k_n \in \mathbb{N} - \{0\}$ and $k_1, k_2, \ldots k_{n-i+1} \in \mathbb{N}$, there exists a monomial ideal I in $R = k[x,y]$ such that $\mathcal{H}_{\mathbf{R/I}} = S'$. Once we prove this fact, the proof of Theorem 4.5.14 become much easier.

Theorem 4.5.12. *Let $R = k[x,y]$ and fix $n, k_n \in \mathbb{N}-\{0\}$ and $k_{n-i+1}, k_{n-i+2}, \ldots, k_{n-1} \in \mathbb{N}$. For each $0 \leq i \leq n-1$, let*

$$I_{(i)} = \langle x^{n-1}y, x^{n-2}y^{1+k_n}, x^{n-3}y^{2+k_n+k_{n-1}}, \ldots, x^{n-1-i}y^{i+\sum_{j=n-i+1}^{n} k_j}\rangle.$$

Then

$$\mathcal{H}_{\mathbf{R/I_{(i)}}} = (1,2,3,\ldots,\underbrace{n,\ldots,n}_{k_n \text{ times}},\underbrace{n-1,\ldots,n-1}_{k_{n-1} \text{ times}},\ldots,\underbrace{n-i+1,\ldots,n-i+1}_{k_{n-i+1} \text{ times}},n-i,n-i,\ldots).$$

Proof. We will prove this by induction on i. Notice that if $i = 0$, then $I_{(0)} = \langle x^{n-1}y\rangle$. By Theorem 4.3.3, we have $\mathcal{H}\big(R/I_{(0)}, i\big) = i+1$ for $0 \leq i < n$ and $\mathcal{H}\big(R/I_{(0)}, j\big) = n$ for all $j \geq n$. Therefore, $\mathcal{H}_{\mathbf{R/I_{(0)}}} = (1,2,3,\ldots,n,n,n,\ldots)$ and so the result holds.

Since the case where $i = 0$ gives us no new information, we will prove that the result holds when $i = 1$ to get a better idea of what is going on. If $i = 1$, then $I_{(1)} = \langle x^{n-1}y, x^{n-2}y^{1+k_n}\rangle$ where $k_n \geq 1$. Since the total degree of the monomial $x^{n-2}y^{1+k_n}$ is $(n-2)+(1+k_n) = n+k_n-1$, it follows that there are no monomials in $I_{(1)}$ of total degree less than n. Hence $\mathcal{H}\big(R/I_{(1)}, i\big) = i+1$ for $0 \leq i \leq n-1$.

We first consider the monomials of total degree j in $I_{(1)}$ where $n \leq j < n+k_n-1$. Since the total degree of $x^{n-2}y^{1+k_n}$ is $n+k_n-1$, then the set of monomials of total degree j in $I_{(1)}$ is

$$\{(x^{n-1}y)(x^a y^b) \mid a+b = j-n\}.$$

Notice that the number of distinct monomials in this set is simply the number of monomials in x and y of total degree $j-n$. By Theorem 2.2.6, there are $j-n+1$ such monomials. Therefore, $\mathcal{H}\big(I_{(1)}, j\big) = j-n+1$ and so

$$\mathcal{H}\big(R/I_{(1)}, j\big) = \mathcal{H}(R,j) - \mathcal{H}\big(I_{(1)}, j\big) = (j+1)-(j-n+1) = n$$

when $n \leq j < n+k_n-1$.

Next, we consider the monomials of total degree r where $r \geq n + k_n - 1$. For a fixed r, we define

$$A_r = \{(x^{n-1}y)(x^a y^b) \mid a + b = r - n\}$$

to be the set of distinct monomials in $I_{(1)}$ of total degree r generated by the monomial $x^{n-1}y$, and

$$B_r = \{(x^{n-2}y^{1+k_n})(x^c y^d) \mid c + d = r - n - k_n + 1\}$$

to be the set of distinct monomials in $I_{(1)}$ of total degree r generated by the monomial $x^{n-2}y^{1+k_n}$. It follows that the monomials in $I_{(1)}$ of total degree r is the set $A_r \cup B_r$. By a well-known fact from set theory, we conclude that

$$\mathcal{H}\big(R/I_{(1)}, r\big) = |A_r \cup B_r| = |A_r| + |B_r| - |A_r \cap B_r|.$$

It is not difficult to see that $|A_r| = r - n + 1$ and $|B_r| = (r - n - k_n + 1) + 1 = r - n - k_n + 2$.

Now we only need to investigate which monomials are in $A_r \cap B_r$. Notice that if $c \geq 1$, then

$$(x^{n-2}y^{1+k_n})(x^c y^d) = (x)(x^{n-2}y^{1+k_n})(x^{c-1}y^d) =$$
$$(x^{n-1}y^{1+k_n})(x^{c-1}y^d) = (x^{n-1}y)(x^{c-1}y^{d+k_n}).$$

Since $x^{c-1}y^{d+k_n}$ has total degree

$$(c-1) + (d + k_n) = (k_n - 1) + (c + d) = (k_n - 1) + (r - n - k_n + 1) = r - n,$$

then it follows that $(x^{n-2}y^{1+k_n})(x^c y^d) \in A_r$ whenever $c \geq 1$. Hence

$$A_r \cap B_r = \{(x^{n-2}y^{1+k_n})(x^c y^d) \mid c + d = r - n - k_n + 1 \text{ and } c \geq 1\}.$$

This means that the only monomial in B_r that is not in A_r is the monomial $x^{n-2}y^{r-n+2}$ and so $|A_r \cap B_r| = (r - n - k_n + 2) - 1 = r - n - k_n + 1$.

Therefore, $\mathcal{H}\big(I_{(1)}, r\big) = (r-n+1)+(r-n-k_n+2)-(r-n-k_n+1) = r-n+2$. We conclude that

$$\mathcal{H}\big(R/I_{(1)}, r\big) = \mathcal{H}(R, r) - \mathcal{H}\big(I_{(1)}, r\big) = (r + 1) - (r - n + 2) = n - 1$$

for all $r \geq n + k_n - 1$, and so

$$\mathcal{H}_{\mathbf{R/I_{(1)}}} = (1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, n-1, n-1, n-1, \ldots).$$

This shows that the result holds when $i = 1$.

Now suppose that the result holds for all $i = m$ where $0 \leq m \leq n - 2$. We will show that it also holds for $m + 1$. By hypothesis, if $i = m$ then

$$I_{(m)} = \langle x^{n-1}y, x^{n-2}y^{1+k_n}, x^{n-3}y^{2+k_n+k_{n-1}}, \ldots, x^{n-1-m}y^{m+\sum_{j=n-m+1}^{n} k_j}\rangle$$

and

$$\mathcal{H}_{\mathbf{R}/\mathbf{I}_{(\mathbf{m})}} = (1,2,3,\ldots,\underbrace{n,\ldots,n}_{k_n \text{ times}},\underbrace{n-1,\ldots,n-1}_{k_{n-1} \text{ times}},\ldots,\underbrace{n-m+1,\ldots,n-m+1}_{k_{n-m+1} \text{ times}},n-m,n-m,\ldots).$$

In particular, notice that if $s \geq n-1+\sum_{j=n-m+1}^{n} k_j$, by hypothesis $\mathcal{H}(R/I_{(m)},s) = n-m$. This means that $\mathcal{H}(I_{(m)},s) = \mathcal{H}(R,s) - \mathcal{H}(R/I_{(m)},s) = (s+1)-(n-m) = s+m-n+1$. We see that for a fixed $s \geq n-1+\sum_{j=n-m+1}^{n} k_j$, the set of distinct monomials in $I_{(m)}$ of total degree s is

$$\{x^{s-1}y, x^{s-2}y^2, \ldots, x^{n-1}y^{s-n+1}, x^{n-2}y^{s-n+2}, x^{n-3}y^{s-n+3}, \ldots, x^{n-1-m}y^{s+m-n+1}\},$$

and there are in fact $s+m-n+1$ such monomials.

Finally, let $i = m+1$ and so

$$I_{(m+1)} = \langle x^{n-1}y, x^{n-2}y^{1+k_n}, x^{n-3}y^{2+k_n+k_{n-1}}, \ldots, x^{n-m-2}y^{m+1+\sum_{j=n-m}^{n} k_j}\rangle.$$

Since $x^{n-m-2}y^{m+1+\sum_{j=n-m}^{n} k_j}$ is a monomial with total degree $n-1+\sum_{j=n-m}^{n} k_j$, then it follows that $\mathcal{H}\big(R/I_{(m)},r\big) = \mathcal{H}\big(R/I_{(m+1)},r\big)$ for all $0 \leq r < n-1+\sum_{j=n-m}^{n} k_j$. Thus we only need to consider the monomials of total degree $t \geq n-1+\sum_{j=n-m}^{n} k_j$. By hypothesis, we define the set of distinct monomials in $I_{(m+1)}$ of total degree t generated by the set

$$\{x^{n-1}y, x^{n-2}y^{1+k_n}, x^{n-3}y^{2+k_n+k_{n-1}}, \ldots, x^{n-1-m}y^{m+\sum_{j=n-m+1}^{n} k_j}\}$$

to be

$$A_t = \{x^{t-1}y, x^{t-2}y^2, \ldots, x^{n-1}y^{t-n+1}, x^{n-2}y^{t-n+2}, x^{n-3}y^{t-n+3}, \ldots, x^{n-1-m}y^{t-n+m+1}\}$$

and there are $t+m-n+1$ such monomials in this set.

Now let

$$B_t = \{(x^{n-m-2}y^{m+1+\sum_{j=n-m}^{n} k_j})(x^c y^d) \mid c+d = t-n-(\sum_{j=n-m}^{n} k_j)+1\}$$

denote the set of distinct monomials in $I_{(m+1)}$ of total degree t generated by the monomial $x^{n-m-2}y^{m+1+\sum_{j=n-m}^{n} k_j}$. Then we have

$$|B_t| = \Big(t-n-(\sum_{j=n-m}^{n} k_j)+1\Big)+1 = t-n-(\sum_{j=n-m}^{n} k_j)+2.$$

Therefore, the set of all monomials in $I_{(m+1)}$ of total degree t is $A_t \cup B_t$.

Notice that if $c \geq 1$, then $(x^{n-m-2}y^{m+1+\sum_{j=n-m}^{n} k_j})(x^c y^d) \in A_t$. Thus the only monomial in B_t that is not in A_t is the monomial $x^{n-m-2}y^{t-n+m+2}$ and so

$$|A_t \cap B_t| = \Big(t-n-(\sum_{j=n-m}^{n} k_j)+2\Big)-1 = t-n-(\sum_{j=n-m}^{n} k_j)+1.$$

Therefore,

$$\mathcal{H}\big(I_{(m+1)}, t\big) = |A_t \cup B_t| = |A_t| + |B_t| - |A_t \cap B_t| =$$

$$(t+m-n+1)+\Big(t-n-(\sum_{j=n-m}^{n} k_j)+2\Big)-\Big(t-n-(\sum_{j=n-m}^{n} k_j)+1\Big) = t+m-n+2.$$

It follows that

$$\mathcal{H}\big(R/I_{(m+1)}, t\big) = \mathcal{H}(R,t)-\mathcal{H}\big(I_{(m+1)}, t\big) = (t+1)-(t+m-n+2) = n-m-1 = n-(m+1)$$

for all $t \geq n-1+\sum_{j=2}^{n} k_j$. Finally, we have

$$\mathcal{H}_{\mathbf{R/I_{(m+1)}}} = (1,2,3,\ldots,\underbrace{n,\ldots,n}_{k_n \text{ times}},\underbrace{n-1,\ldots,n-1}_{k_{n-1} \text{ times}},\ldots,\underbrace{n-m,\ldots,n-m}_{k_{n-m} \text{ times}},n-m-1,n-m-1,\ldots),$$

as was desired. We conclude that the result holds when $0 \leq i \leq n-1$. $\square$

If S is an infinite sequence that cannot be written in the form given in Theorem 4.5.12, then S is not a Hilbert sequence by Theorem 4.5.5. The most important consequence that follows this fact and Theorem 4.5.12 is that every infinite sequence $(1,2,3,\ldots,n,n_1,n_2,n_3,n_4,\ldots)$ is a Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ for some monomial ideal I in $R = k[x,y]$ when $n \geq n_1 \geq n_2, \geq n_3, \geq n_4, \geq \cdots$. Therefore, we now know exactly what every infinite Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ in R looks like.

Example 4.5.13. Theorem 4.5.12 also shows us how we can find a monomial ideal I such that $\mathcal{H}_{\mathbf{R/I}}$ is any infinite sequence of the form $(1,2,3,\ldots,n,n_1,n_2,n_3,n_4,\ldots)$ where $n \geq n_1 \geq n_2 \geq n_3 \geq n_4 \geq \cdots$. For example, suppose that $S = (1,2,1,1,1,1,\ldots)$ and we want to find an I such that $\mathcal{H}_{\mathbf{R/I}} = S$. Theorem 4.5.12 tells us that since $n = 2$ and $n_2 = 1$, then if $I = \langle xy, y^2\rangle$, then $\mathcal{H}_{\mathbf{R/I}} = S$. We saw in Example 3.1.2 that if $I' = \langle x^2, xy\rangle$, then $\mathcal{H}_{\mathbf{R/I'}} = (1,2,1,1,1,1,\ldots)$. Notice that if we interchange x and y in each generator in I', this will not affect the number of monomials of any given total degree. Also note that if we interchange x and y in I' then we get I. Therefore, the result of Theorem 4.5.12 holds. This example also shows that the ideal I given in Theorem 4.5.12 is not the only ideal with the Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$. There may exist other monomial ideals I' such that $\mathcal{H}_{\mathbf{R/I}} = \mathcal{H}_{\mathbf{R/I'}}$. $\lozenge$

We are finally ready to show that if S_i is the set of all sequences of length i given in Theorem 4.5.9, then for every sequence $S \in S_i$, there exists a monomial ideal I in $R = k[x,y]$ such that $\mathcal{H}_{\mathbf{R/I}} = S$.

Theorem 4.5.14. *Let $R = k[x,y]$ and suppose that*

$$I = \langle x^{n-1}y, x^{n-2}y^{1+k_n}, x^{n-3}y^{2+k_n+k_{n-1}}, \ldots, x^{n-1-i}y^{i+\sum_{j=n-i+1}^{n} k_j},$$
$$\ldots, x^2y^{n-3+\sum_{j=4}^{n} k_j}, xy^{n-2+\sum_{j=3}^{n} k_j}, y^{n-1+\sum_{j=2}^{n} k_j}, x^{n-1+\sum_{j=1}^{n} k_j}\rangle$$

for some $n, k_n \in \mathbb{N} - \{0\}$ and $k_1, k_2, \ldots, k_{n-1} \in \mathbb{N}$. Then

$$\mathcal{H}_{\mathbf{R/I}} = (1,2,3,\ldots,\underbrace{n,\ldots,n}_{k_n \text{ times}},\underbrace{n-1,\ldots,n-1}_{k_{n-1} \text{ times}},\ldots,\underbrace{4,\ldots,4}_{k_4 \text{ times}},\underbrace{3,\ldots,3}_{k_3 \text{ times}},\underbrace{2,\ldots,2}_{k_2 \text{ times}},\underbrace{1,\ldots,1}_{k_1 \text{ times}},0,0,\ldots).$$

Proof. Since the total degree of the monomial $x^{n-1+\sum_{j=1}^{n} k_j}$ is $n-1+\sum_{j=1}^{n} k_j$, then it follows that for all $0 \leq m < n-1+\sum_{j=1}^{n} k_j$, we have

$$\mathcal{H}(R/I, m) = \mathcal{H}\big(R/I_{(n-1)}, m\big)$$

where $I_{(n-1)}$ is the ideal given in Theorem 4.5.12 when $i = n-1$. As we saw in the proof of Theorem 4.5.12, if $p = n-1+\sum_{j=1}^{n} k_j$, then the set of distinct monomials in I of degree p generated by the set

$$\{x^{n-1}y, x^{n-2}y^{1+k_n}, x^{n-3}y^{2+k_n+k_{n-1}}, \ldots, x^2 y^{n-3+\sum_{j=4}^{n} k_j}, xy^{n-2+\sum_{j=3}^{n} k_j}, y^{n-1+\sum_{j=2}^{n} k_j}\}$$

is

$$A_p = \{x^{p-1}y, x^{p-2}y^2, \ldots, x^2y^{p-2}, xy^{p-1}, y^p\}.$$

Notice that $|A_p| = p$. Also note that x^p is the only monomial of total degree p in R that is not in the set A. However, since the monomial $x^p = x^{n-1+\sum_{j=1}^{n} k_j}$ is a generator for I, we conclude that all monomials of total degree p are in I. Thus $\mathcal{H}(R/I, p) = 0$ and by Theorem 4.3.1, $\mathcal{H}(R/I, q) = 0$ for all $q \geq p$.

Therefore,

$$\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, \ldots, \underbrace{4, \ldots, 4}_{k_4 \text{ times}}, \underbrace{3, \ldots, 3}_{k_3 \text{ times}}, \underbrace{2, \ldots, 2}_{k_2 \text{ times}}, \underbrace{1, \ldots, 1}_{k_1 \text{ times}}, 0, 0, \ldots).$$

□

Theorem 4.5.14 shows us that every finite sequence of the form $(1, 2, 3, \ldots, n, n_1, n_2, \ldots, n_{i-n})$ is a Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ for some monomial ideal I when $n \geq n_1 \geq n_2 \geq \cdots \geq n_{i-n} > 0$ and $1 \leq n \leq i$. Also note that by Theorem 4.5.5, these are the only possible finite Hilbert sequences. Therefore, Theorem 4.5.9 tells us that there are exactly 2^{i-1} finite Hilbert sequences of length i.

Example 4.5.15. Let $S = (1, 2, 3, 2, 1)$ and suppose we want to find a monomial ideal I such that $S = \mathcal{H}_{\mathbf{R/I}}$. Theorem 4.5.14 tells us that if $I = \langle x^2y, xy^2, y^4, x^5\rangle$, then $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 2, 1)$. We see that there are no monomials in I of total degree 0, 1, or 2, and so $\mathcal{H}(R/I, 0) = 1$, $\mathcal{H}(R/I, 1) = 2$, and $\mathcal{H}(R/I, 2) = 3$. We also see that the set of monomials in I of total degree 3 is $\{x^2y, xy^2\}$ and so $\mathcal{H}(I, 3) = 2$ which implies that $\mathcal{H}(R/I, 3) = \mathcal{H}(R, 3) - \mathcal{H}(I, 3) = (3+1) - 2 = 2$. Similarly, the set of monomials in I of total degree 4 is $\{x^3y, x^2y^2, xy^3, y^4\}$ and so $\mathcal{H}(I, 4) = 4$. Thus $\mathcal{H}(R/I, 4) = \mathcal{H}(R, 4) - \mathcal{H}(I, 4) = (4+1) - 4 = 1$. Finally, we note that set of monomials of total degree 5 in I is $\{x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\}$ and this is the set of all possible monomials of total degree 5 in R. Hence there are no monomials of total degree 5 in R/I and so $\mathcal{H}(R/I, 5) = 0$. We conclude that $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 2, 1)$, and so Theorem 4.5.14 holds. ♢

We have now found every possible Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ when I is a monomial ideal in $R = k[x, y]$. Since $\mathcal{H}_{\mathbf{I}} = \mathcal{H}_{\mathbf{R}} - \mathcal{H}_{\mathbf{R/I}}$, we have also found every possible Hilbert sequence $\mathcal{H}_{\mathbf{I}}$. However, we have shown even more than this. We have actually found all Hilbert sequences for any homogeneous ideal in $k[x, y]$. This brings us to our main result.

Theorem 4.5.16. *$\mathcal{H}_{\mathbf{R/I}}$ is the Hilbert sequence of some homogeneous ideal in I if and only if $\mathcal{H}_{\mathbf{R/I}}$ has the form*

$$(1, 2, 3, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{k_{n-1} \text{ times}}, \ldots, \underbrace{n-i+1, \ldots, n-i+1}_{k_{n-i+1} \text{ times}}, n-i, n-i, \ldots)$$

for some $n, k_n \in \mathbb{N} - \{0\}$, $k_1, k_2, \ldots, k_{n-i+1} \in \mathbb{N}$, and $0 \leq i \leq n$.

Proof. Corollary 4.5.7 and Theorems 4.5.12 and 4.5.14 prove this for all monomial ideals in $k[x, y]$. It follows from Proposition 1.0.1 that the theorem holds for any homogeneous ideal I in $k[x, y]$. □

Notice that in Theorem 4.5.16, if $0 \leq i < n$ then $\mathcal{H}_{\mathbf{R/I}}$ is an infinite sequence, while if $i = n$ then $\mathcal{H}_{\mathbf{R/I}}$ is a finite sequence. We now know every possible Hilbert sequence $\mathcal{H}_{\mathbf{R/I}}$ in $k[x, y]$ when I is a homogeneous ideal.

5

Open Questions

I now present the reader with a list of questions that I encountered while working on this senior project. I find these questions to be engaging, but I did not have enough time during the past year to pursue answers to any of them. Naturally, many other questions arose as my project progressed, but the questions that I list are the ones that I find to be the most promising. Perhaps someday a devoted reader will be able to provide me with an answer to some, if not all of them.

1. After computing hundreds of examples, I am quite convinced that if $R = k[x, y, z]$ and I is any monomial ideal in R, then once the sequence $\mathcal{H}_{\mathbf{R/I}}$ begins to decrease, it can never increase again. We were able to prove in Section 4.5 that this is true when $R = k[x, y]$, and so is it possible to generalize the proofs of Theorems 4.5.3 and 4.5.5 for three indeterminates instead of two?

2. In fact, I would be willing to bet that if $R = k[x_1, \ldots, x_n]$ and I is any monomial ideal in R, then if the sequence $\mathcal{H}_{\mathbf{R/I}}$ starts decreasing, it will never increase again. Can this be proven or does there exist a counterexample?

3. In Section 4.5, we saw that once we proved Theorem 4.5.5, we were then able to count the number of finite Hilbert sequences of a given length. How many finite Hilbert sequences of a given length are there in $k[x, y, z]$? Note that $k[x, y, z]/\langle z \rangle \cong k[x, y]$ and so every Hilbert sequence in $k[x, y]$ is also a Hilbert sequence in $k[x, y, z]$. What other types of finite Hilbert sequences are in $k[x, y, z]$ that are not in Hilbert sequences in $k[x, y]$? How about the finite Hilbert sequences in $k[x_1, \ldots, x_n]$?

4. We saw in Example 4.5.13 that both of the ideals $I = \langle xy, y^2 \rangle$ and $I' = \langle x^2, xy \rangle$ had the same Hilbert sequence $(1, 2, 1, 1, 1, 1, \ldots)$. However, this makes sense

because we can get one ideal from the other by interchanging x and y in the generators. We also saw in Example 4.5.15 that if $I = \langle x^2y, xy^2, y^4, x^5\rangle$, then $\mathcal{H}_{\mathbf{R/I}} = (1, 2, 3, 2, 1)$. It is not difficult to show that if $I' = \langle x^3, y^3\rangle$, then $\mathcal{H}_{\mathbf{R/I'}} = (1, 2, 3, 2, 1)$ as well. In this case, we cannot get I by interchanging x and y in the generators of I'. In fact, I' has the fewest number of generators possible in order to obtain the sequence $(1, 2, 3, 2, 1)$ (for example, if I' is generated by one monomial, then $\mathcal{H}_{\mathbf{R/I'}}$ is infinite). Thus we can think of I' as the ideal generated by the fewest number of monomials such that $\mathcal{H}_{\mathbf{R/I'}} = (1, 2, 3, 2, 1)$. Given any Hilbert sequence S in $k[x, y]$, how does the ideal I' generated by the fewest number of monomials such that $\mathcal{H}_{\mathbf{R/I'}}$ compare to the ideal given to us in Theorem 4.5.12 or 4.5.14? Are there any relationships between all ideals who have the same Hilbert sequence?

5. In Section 4.3, we were able to prove some results for $k[x_1, \ldots, x_n]$. What other results can be proved in full generality?

6. What kind of Hilbert sequences are possible in $k[x, y]$ when the ideals are generated by a set of monomials of the same total degree? For example, if we look at all ideals that are generated by a set of monomials of total degree 2, it is not difficult to see that there are 6 such possible ideals and we get the following 4 different Hilbert sequences:

   (a) $(1, 2, 2, 2, \ldots)$ if the ideal is generated by a single monomial of total degree 2

   (b) $(1, 2, 1, 1, \ldots)$ if the ideal is $\langle x^2, xy\rangle$, or $\langle y^2, xy\rangle$

   (c) $(1, 2, 1)$ if the ideal is $\langle x^2, y^2\rangle$

   (d) $(1, 2)$ if the ideal is generated by all monomials of total degree 2.

   What kind of relationship is there between the number of generators in an ideal of this form and the corresponding Hilbert sequence? How many such Hilbert sequences are there for all possible ideals generated by a set of monomials of total degree i?

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