

Clifford Algebra Representations

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Abstract

The Clifford algebras are number systems that expand the real numbers by adjoining elements that square roots of negative one to the real numbers, such as i in the complex numbers. In fact, they constitute a way to generalize the complex numbers, as well as the quaternions. The presence of multiple roots of -1 necessitates rules for multiplying them, which we call anticommutativity. For example, in the quaternions where i and j are both roots of -1 , i and j anticommute, so $ij = -ji$. As the number of roots of -1 increase, the number system grows accordingly more difficult to work with.

Because Clifford algebras are vector spaces, a wide variety of results from Linear Algebra can be applied to them. In particular, we can view the individual elements as matrices using techniques from Representation Theory. Then we will shrink these matrices using certain linear transformations known as projections. Many properties of these projections will then be explored, in particular identifying which projections can be used to produce identical representations and counting them.

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Dedication

To my family, for all they have done.

Acknowledgments

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1

An Introduction to Clifford Algebras

1.1 Historical Information

In treating the Clifford algebras, we first turn to their namesake, William Kingdon Clifford. Born in 1845, Clifford was a British mathematician and avowed atheist. He died in 1879 of tuberculosis, one of many historical figures to die too young. Like many great mathematicians, he showed great promise from a young age, excelled in his studies (at Cambridge) and taught at a prestigious university, in his case University College in London. In 1875 he married Ethel Lucy, who went on to become a well known author after his death. They had two children who were infants at the time of his death. Although his mathematical contributions are valued to this day, Clifford is perhaps best known for his advocacy of evolution and atheism. We can see how he applied the standards of mathematics to philosophy from the following passage found in essay *The Ethics of Belief*: "It is wrong always, everywhere, and for anyone, to believe anything upon insufficient evidence."

W. K. Clifford's mathematical contributions are mostly found in the field of geometry. Much of his work was based on the previous contributions of William Hamilton, in particular his quaternions. His studies in abstract spaces have proven valuable through the

years. Some of his ideas are precursors to the geometry of general relativity. Today, he is best known for his eponymous algebras. (All information taken from [1, Introduction]).

1.2 Objectives

Clifford algebras can be thought of as directed number systems. They are a generalization of the complex numbers \mathbb{C} and the quaternions \mathbb{H} . The quaternions are numbers of the form $a + bi + cj + dk$ where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$. Invented by Hamilton, they can be used to represent rotations in three-dimensional space. In order to arrive at the Clifford algebras, we adjoin additional square roots of -1 while maintaining the rule $ij = -ji$ for all such roots. Such a number system has applications in geometry of higher dimensions and theoretical physics, as well as areas as far afield as coding theory.

As algebras, the Clifford algebras are both rings and vector spaces. This allows us to build matrices from the left action of the Clifford algebra on itself that have identical properties to the algebra itself. These matrices constitute a representation of the Clifford algebra. The standard method for describing such matrices does not always find the smallest matrices with these properties and so is not always a minimal representation. Using objects called spinors, a standard treatment for finding the minimal representation was developed. Dimakis has demonstrated that minimal representations can also be found using linear transformations called projections which map from a vector space to a subspace [1, Dimakis].

In this project, we begin by developing the necessary background to follow the approach outlined by Dimakis. It is assumed that the reader has had first courses in linear and abstract algebra. This background focuses on defining $Cl(n)$, the Clifford algebra with n adjoined square roots of -1 and developing its properties as both vector spaces and rings. In particular, we demonstrate the standard matrix representation of Clifford algebras and define projections.

The next chapter is devoted to deriving and explaining Dimakis's method. In order to do so, we first show how to construct projections out of elements which square to 1. We then explain how such projections split the algebra into two identical subspaces which are also algebras. Next, we demonstrate how to represent $Cl(3)$ and $Cl(4)$ using projections and describe how they function inside of the algebra. Following these examples, we return to theory in order to explain how multiple projection elements interact. Afterwards, we find the representations of all up to $Cl(8)$ and outline a one method for constructing a minimal representation of $Cl(n)$.

In our final chapter, we explore different ways of constructing our minimal representations. In doing so, we find all of the distinct constructions for the minimal representation of all Clifford algebras up to $Cl(16)$. We then search these constructions for patterns and develop some preliminaries towards finding a formula to count the number of such distinct constructions given an arbitrary number adjoined roots.

2

Preliminaries

2.1 Defining the Clifford Algebras

We begin by defining terms necessary to understand what a Clifford algebra is. Perhaps the most basic word here to define is in the very name of the subject: algebra.

Definition 2.1.1. Let F be a field. An *algebra* A over F is a ring A such that

1. $(A, +)$ is a vector space over F
2. For $f \in F$ and $a, b \in A$, we have $f(ab) = (fa)b = a(fb)$.

△

An algebra that is a finite dimensional vector space over F is called a *finite dimensional algebra*. It is important to note that sometimes algebras are constructed over rings instead of fields, but this definition is more directly applicable to the work at hand, as well as obviating the need to go into additional depth describing aspects of algebras over rings that will not be used in this project.

As is the case for many objects in algebra, there are numerous equivalent definitions for Clifford algebras. We choose the following one for the ease of computation that it offers ;

Definition 2.1.2. The Clifford algebra with n generators $Cl(n)$ is defined as the algebra over the real numbers \mathbb{R} with n multiplicative generators $\gamma_1, \gamma_2, \dots, \gamma_n$ called *Clifford generators* which satisfy the relations

1. $\gamma_i^2 = -1$ for $i = 1, \dots, n$,
2. $\gamma_i \gamma_j = -\gamma_j \gamma_i$ for $i \neq j$.

When one element is the product of multiple Clifford generators, we will write its subscripts together (e.g. $\gamma_1 \gamma_2 = \gamma_{12}$ and $\gamma_4 \gamma_2 \gamma_3 = \gamma_{423}$). △

Just as our definition of algebra is not entirely inclusive, not all Clifford algebras match this definition. Sometimes they can be over different fields, such as \mathbb{C} . In other instances, some or all of the Clifford generators square to 1 instead of -1 . Because a more general description requires additional terminology and concepts gratuitous to our needs, we choose this narrower definition.

There is an obvious choice for the canonical basis of $Cl(n)$. From Definition 2.1.2 (2), the ordering of Clifford generators is arbitrary up to sign. By Definition 2.1.2 (1), any Clifford generator present n times within a single word of Clifford generators need only appear $n \bmod 2$ times, while introducing a scalar multiple. Therefore, we can describe the canonical basis of $Cl(n)$ in terms of the products of Clifford generators as

$$\gamma_{i_1} \cdots \gamma_{i_k}, i_1 < i_2 < \cdots < i_k, k = 0, 1, \dots, n.$$

We order the basis elements first by the number of Clifford generators contained in the basis element and then by the sum of the various i_k . For example, γ_3 comes before γ_{12} and before γ_4 . In order to find the dimension of $Cl(n)$, let $S = \{1, \dots, n\}$. Then any subset of S , will correspond to a basis element of $Cl(n)$. For example, $\{2, 4, 7\}$ is a subset of S where $n = 7$. It then corresponds to γ_{247} and visa versa. We then have a bijective function

mapping the canonical basis of $Cl(n)$ to the power set $P(S)$, so

$$\dim Cl(n) = |P(S)| = 2^n.$$

Clearly $Cl(0) = \mathbb{R}$. The Clifford algebras with one and two generators are also familiar algebras.

Example 2.1.3. The canonical basis of $Cl(1)$ is the scalar 1 and the Clifford generator γ_1 . Therefore, every element of $Cl(1)$ can then be described as $a + b\gamma_1$, with $a, b \in \mathbb{R}$ such that for two elements in $Cl(1)$,

$$(a_1 + b_1\gamma_1) + (a_2 + b_2\gamma_1) = (a_1 + a_2) + (b_1 + b_2)\gamma_1,$$

and

$$\begin{aligned} (a_1 + b_1\gamma_1)(a_2 + b_2\gamma_1) &= a_1a_2 + a_1b_2\gamma_1 + a_2b_1\gamma_1 + b_1b_2(\gamma_1)^2 \\ &= a_1a_2 + (a_1b_2 + a_2b_1)\gamma_1 + b_1b_2(-1) \\ &= a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)\gamma_1. \end{aligned}$$

and replacing γ_1 with i , we see that $Cl(1) = \mathbb{C}$. ◇

Example 2.1.4. The canonical basis for $Cl(2)$ consists of 1, γ_1 , γ_2 and γ_{12} (note that $\gamma_{21} = -\gamma_{12}$ by definition). In \mathbb{H} , the Quaternions, we have the relationship that $ij = k = -ji$. As

$$(\gamma_{12})^2 = (-\gamma_2\gamma_1)(\gamma_1\gamma_2) = -(\gamma_2)(\gamma_1\gamma_1)(\gamma_2) = (\gamma_2\gamma_2) = -1,$$

we can think of γ_{12} as k and $Cl(2) = \mathbb{H}$. ◇

2.2 Algebraic Properties

Recall that an algebra is both a ring and a vector space. As rings, Clifford algebras have many useful properties. The following one will prove quite useful.

Definition 2.2.1. A subring I of a ring R is a *left ideal* if for all $r \in R$ and $x \in I$ we have $rx \in I$.

△

This will prove useful in maintaining ring properties on certain vector spaces.

Much of what we do with Clifford algebras depends on the fact that they are vector spaces.

Definition 2.2.2. In any algebra A , the operation of left multiplication $l_a : A \rightarrow A$ by an element $a \in A$ is defined as $l_a(b) = ab$.

△

We will show l_a is a linear transformation from A to A when viewing A as a vector space. In order to verify this, recall that for vector spaces V_1 and V_2 over field F , the function $T : V_1 \rightarrow V_2$ is a linear transformation when

$$T(b + c) = T(b) + T(c), \quad \forall b, c \in V_1$$

and

$$T(fb) = fT(b) \quad \forall b \in V_1, f \in F.$$

As $l_a(b + c) = a(b + c) = ab + ac$ and $l_a(fb) = a(fb) = f(ab)$, we know l_a is a linear transformation.

Next, we describe one important type of linear transformation found in $Cl(n)$.

Definition 2.2.3. The linear transformation $T \in \text{End}(V)$ (T is an endomorphism) is an *involution* if $T^2 = I$, the identity transformation.

△

Clifford algebras can have many involutions aside from 1 and -1 ,

Example 2.2.4. Let $\omega = \gamma_{123}$. Then l_ω is a linear transformation. We compute

$$\begin{aligned} (l_\omega)^2(1) &= \gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\ &= -\gamma_1\gamma_2\gamma_1\gamma_3\gamma_2\gamma_3 \\ &= \gamma_1\gamma_1\gamma_2\gamma_3\gamma_2\gamma_3 \\ &= -\gamma_1\gamma_1\gamma_2\gamma_2\gamma_3\gamma_3 = (-1)(-1)(-1)(-1) = 1. \end{aligned}$$

Therefore $l_{\gamma_{123}}$ is an involution. We can say that γ_{123} is an involution as well. \diamond

Each element a in the algebra A corresponds to a unique left transformation l_a . In order to perform computations with these linear transformations, we need to describe this correspondence.

Definition 2.2.5. A *representation* of A is an algebra homomorphism $\rho : A \rightarrow \text{End}(V)$ from the algebra A to the endomorphisms of V .

For $a \in A$, we say a is represented by matrix $\rho(a)$. We can also say that A is represented by $n \times n$ matrices where $\dim V = n$. \triangle

When we construct a representation, it will map each element to the matrix corresponding to that element's linear transformation. For example $Cl(0) \cong \mathbb{R}$, so for $a \in Cl(0)$ the corresponding linear transformation is the one by one matrix $[a]$. Additionally, \mathbb{C} can be represented by 2×2 matrices where $a + bi \in \mathbb{C}$ is represented by

$$\begin{bmatrix} +a & -b \\ +b & +a \end{bmatrix}.$$

This is the matrix for the linear transformation corresponding to complex multiplication by $a + b\gamma_1$ with respect to the standard basis $\{1, \gamma_1\}$ of $Cl(1)$. Because $a, b \in \mathbb{R}$, we see that

$$\left\{ \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \right\}$$

forms a basis for the linear transformations corresponding to elements in $Cl(1)$. As these linear transformations exhibit all of the additive and multiplicative properties of $Cl(1)$,

they are isomorphic to $Cl(1)$. Matrix representations such as these can be constructed for $Cl(n)$ by generalizing the process outlined in the following example.

Example 2.2.6. We will construct the matrix form of the linear transformation corresponding to l_a for $a \in Cl(2)$. Recall

$$a = a_1 + a_2\gamma_1 + a_3\gamma_2 + a_4\gamma_{12}, \quad a_i \in \mathbb{R}. \quad (2.2.1)$$

We can then describe l_a using the linear transformations corresponding to the basis elements. Let b_i be the i th element in the canonical basis of $Cl(2)$. First we will compute the linear transformation corresponding to b , a basis element of $Cl(2)$.

We define the ij th entry of the matrix corresponding to l_b to be the coefficient a_j for $a = bb_i$. We can then have the following representation:

$$\begin{aligned} 1 &\mapsto \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}, & \gamma_1 &\mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{bmatrix} \\ \gamma_2 &\mapsto \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, & \gamma_{12} &\mapsto \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

By adding together these matrix representations of basis elements, each multiplied by the appropriate scalar, we construct the matrix l_a as

$$a \mapsto \begin{bmatrix} +a_1 & -a_2 & -a_3 & -a_4 \\ +a_2 & +a_1 & -a_4 & +a_3 \\ +a_3 & +a_4 & +a_1 & -a_2 \\ +a_4 & -a_3 & +a_2 & +a_1 \end{bmatrix}.$$

◇

Because this method of representation is guaranteed to be faithful (injective), we can identify $a \in Cl(n)$ with l_a . This identification will often be made implicitly. It also happens in the case of $Cl(2)$ that this representation is minimal (*i.e.*, no smaller such representation

can be constructed). That this representation of $Cl(2)$ is minimal does not guarantee that this process will always produce a minimal representation.

2.3 Projections on Vector Spaces

In this project, we intend to demonstrate how using projections one can meaningfully reduce the amount of information necessary to represent a Clifford algebra. In order to do this, we must first define projections and accompanying terminology. The following definition is necessary in this endeavor.

Definition 2.3.1. Let U and V be vector spaces over the same field F . We define their *direct sum* to be the vector space W (denoted by $U \oplus V$) whose elements are all the ordered pairs $\langle u, v \rangle$ with $u \in U$ and $v \in V$, with the linear operations defined by

$$a\langle u_1, v_1 \rangle + b\langle u_2, v_2 \rangle = \langle au_1 + bu_2, av_1 + bv_2 \rangle, \quad a, b \in F.$$

△

The following well known result about direct sums will be applied in dealing with projections. Its proof is omitted but can be found in [5, Section 18].

Lemma 2.3.2. *The following are equivalent*

1. $W = U \oplus V$.
2. $U \cap V = 0$ and for all $w \in W$, there exists $u \in U$ and $v \in V$ such that $u + v = w$ ($U + V = W$).

Now we have the appropriate background to treat the subject at hand.

Definition 2.3.3. Let W, U and V be vector spaces. If W is the direct sum of U and V , so that every $w \in W$ can be uniquely written in the form $w = u + v$, where $u \in U$ and $v \in V$, the *projection* onto U along V is the linear transformation P defined by $Pw = u$. △

Under the appropriate basis, the matrix form of a projection should be diagonal where each entry is either 1 or 0. This definition provides no way of finding projections, nor does it tell us much about them. Instead, we turn to an important result about projections to guide us. First, however, certain terms must be defined.

Definition 2.3.4. An element $a \in A$ is *idempotent* if $a^2 = a$. △

Now we can state the following:

Lemma 2.3.5. *A linear transformation P is a projection onto some subspace if and only if it is idempotent.*

Proof. If P is a projection from W onto U along V where $w \in W$, $u \in U$ and $v \in V$ such that $w = u + v$, then the decomposition of w is $w = u + 0$, so

$$P^2w = PPw = Pu = u = Pw.$$

Conversely, suppose $P^2 = P$. Let V be the set of all vectors $w \in W$ such that $Pw = 0$; let U be the set of all vectors w for which $Pw = w$. Clearly, U and V are subspaces of W .

If $w \in U$, then $Pw = w$; if $w \in V$, then $Pw = 0$. Hence, if $w \in U$ and $w \in V$, then $w = 0$. Therefore $U \cap V = 0$, and they are disjoint. For an arbitrary w , we have

$$w = Pw + (1 - P)w.$$

Let $u = Pw$ and $v = (1 - P)w$. Then $Pu = P^2w = Pw = u$ and $Pv = P(1 - P)w = Pw - P^2w = 0$, so $u \in U$ and $v \in V$. Then by Lemma 2.3.2, $W = U \oplus V$ and P is the projection onto U along V . □

This proof is based on the one found in [5, Section 41].

From this, we also derive the following corollary which will prove quite useful.

Corollary 2.3.6. *If P is the projection onto U along V , then $1 - P$ is the projection onto V along U .*

While we will not prove the entire corollary, it is worth observing that $1-P$ is idempotent as

$$(1 - P)^2 = (1 - P)(1 - P) = 1 - 2P + P^2 = 1 - P.$$

Together, the subspaces projected onto by P and $1 - P$ constitute the entire vector space.

3

Minimal Representations of $Cl(n)$

3.1 Reduced Representations Using Projections

We are now prepared to confront the central problem of this project. Using the procedure outlined in Example 2.2.6, we can represent elements in $Cl(n)$ using $2^n \times 2^n$ matrices. However, if we were able to find projections that split the algebra into subspaces, then the matrices corresponding to these subspaces would contain fewer entries. We begin by searching for projection elements.

Lemma 3.1.1. *Let $\omega \in Cl(n)$ such that ω is an involution and ω is orthogonal to 1. Then $\frac{1+\omega}{2}$ and its complementary projection $\frac{1-\omega}{2}$ split $Cl(n)$ into two subspaces of equal dimension.*

Proof. First, we compute

$$\left(\frac{1+\omega}{2}\right)^2 = \frac{1+2\omega+(\omega)^2}{4} = \frac{1+2(\omega)+1}{4} = \frac{1+\omega}{2}.$$

Therefore, by Lemma 2.3.5, $\frac{1+\omega}{2}$ is a projection. As all projections are diagonalizable and the identity matrix is diagonal, we also know that the matrix representation of ω is diagonalizable.

Let A be the standard matrix representation of ω where v is an eigenvector and λ an eigenvalue of ω (these exist in \mathbb{C}). Then

$$v = 1v = A^2v = \lambda^2v$$

and as $v \neq 0$, we find $\lambda^2 = 1$ so λ equals 1 or -1 . When we diagonalize A , its diagonal entries will all be 1 and -1 .

Because ω is orthogonal to 1, ωa is orthogonal to a , for any a in the canonical basis. Therefore A 's diagonal entries are zero so the trace of A is zero (recall the trace is the sum of all diagonal entries). It is well known that trace is invariant between similar matrices, so when we diagonalize A , its trace will still be zero. This means the diagonal must contain an equal number of 1's and -1 's. By reordering the basis, we can put all of the 1's in the upper-left quadrant and all of the -1 's in the lower-right quadrant. After adding the identity matrix, then dividing by the scalar 2, we are left with 1's in the upper-left quadrant diagonal and 0's throughout the rest of the matrix. This is the matrix representation of our projection element, which clearly splits the algebra in half. \square

Note that were $\omega^2 = -1$, its eigenvalues would be i and $-i$, and it would not diagonalize over \mathbb{R} . By Corollary 2.3.6,

$$1 - \frac{1 + \omega}{2} = \frac{1 - \omega}{2}$$

is the complementary projection to $\frac{1+\omega}{2}$. Although these projections split the algebra into two subspaces, there is no guarantee that these subspaces are closed under multiplication. Therefore, we need the following result.

Lemma 3.1.2. *The subspace $Cl(n)\frac{1+\omega}{2}$ projected onto by $\frac{1+\omega}{2}$ is a left ideal of $Cl(n)$.*

Proof. Let $a \in Cl(n)$ and $b \in Cl(n)\frac{1+\omega}{2}$. Then there exists $c \in Cl(n)$ such that $b = c\frac{1+\omega}{2}$.

Then

$$ab = ac\frac{1+\omega}{2} \in Cl(n)\frac{1+\omega}{2}$$

as $ac \in Cl(n)$ by closure of rings. Thus $Cl(n)^{\frac{1+\omega}{2}}$ is a left ideal. \square

Although any element in $Cl(n)$ can be used to form an ideal, because $\frac{1+\omega}{2}$ splits the algebra, the ideal it forms is a non-trivial subalgebra, though it does not necessarily contain a multiplicative identity.

3.2 Representing $Cl(3)$ Minimally

The standard matrix representation of $Cl(3)$ is as follows:

$$\begin{aligned}
 1 &\mapsto \begin{bmatrix} +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix}, \gamma_1 \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \end{bmatrix}, \\
 \\
 \gamma_2 &\mapsto \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \gamma_3 \mapsto \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \end{bmatrix}, \\
 \\
 \gamma_{12} &\mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}, \gamma_{13} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$\gamma_{23} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \gamma_{123} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that elements of $Cl(3)$ can be represented as eight by eight matrices. However, because $(\gamma_{123})^2 = 1$, as shown in Example 2.2.4, we know γ_{123} is an involution and by Lemma 3.1.1 we have the projection $\frac{1+\gamma_{123}}{2}$. We now know that $\frac{1+\gamma_{123}}{2}$, which we will call P^+ , projects onto half of $Cl(3)$ and its complementary projection $\frac{1-\gamma_{123}}{2}$, or P^- , projects onto the other half. However, we don't know how they act on the algebra in order to accomplish this.

Because γ_{123} is an involution, we can factor it out of 1. Then we are left with

$$\frac{1 + \gamma_{123}}{2} = \gamma_{123} \frac{\gamma_{123} + 1}{2} = \gamma_{123} \frac{1 + \gamma_{123}}{2}. \quad (3.2.1)$$

This relationship will allow us to describe the nature of the split.

We compute the following equalities:

$$\begin{aligned} P^+ &= \gamma_{123} P^+ \\ \gamma_1 P^+ &= \gamma_1 \gamma_{123} P^+ = -\gamma_{23} P^+ \\ \gamma_2 P^+ &= \gamma_2 \gamma_{123} P^+ = \gamma_{13} P^+ \\ \gamma_3 P^+ &= \gamma_3 \gamma_{123} P^+ = -\gamma_{12} P^+. \end{aligned}$$

Then, within the ideal $Cl(3)P^+$,

$$1 \sim \gamma_{123}, \gamma_1 \sim -\gamma_{23}, \gamma_2 \sim \gamma_{13}, \text{ and } \gamma_3 \sim -\gamma_{12},$$

which is to say they are identified under the homomorphism $Cl(3) \mapsto Cl(3)P^+$. Clearly, P^+ , $\gamma_2 P^+$, $\gamma_1 P^+$ and $\gamma_3 P^+$ form a basis of the ideal $Cl(3)P^+$. Inside of $Cl(3)P^+$, we

see P^+ is the multiplicative identity because γ_{123} resides in the center of $Cl(3)$, so P^+ commutes with every other element as well. Calling this identity element 1, these basis elements can be represented as follows:

$$P^+ \leftrightarrow \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}, \gamma_2 P^+ \leftrightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{bmatrix}$$

$$\gamma_1 P^+ \leftrightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \gamma_3 P^+ \leftrightarrow \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{bmatrix}$$

Comparing these matrices to the matrix representation of $Cl(2)$ (the quaternions) in Example 2.2.6, we see that they are identical, with P^+ corresponding to 1, etc. Therefore

$$Cl(3)P^+ \cong \mathbb{H}.$$

Consequently, we can represent every element in $Cl(3)$ of the form cP^+ . Now we try to represent the remaining elements. We begin by factoring $-\gamma_{123}$ from P^- , and compute

$$P^- = \frac{1 - \gamma_{123}}{2} = -\gamma_{123} \frac{-\gamma_{123} + 1}{2} = -\gamma_{123} P^-.$$

From this relation, we can compute that $1 \sim -\gamma_{123}$, $\gamma_1 \sim \gamma_{23}$, $\gamma_2 \sim -\gamma_{13}$ and $\gamma_3 \sim \gamma_{12}$ inside of $Cl(3)P^-$. By letting P^- be the identity, we can again reconstruct $Cl(2)$ as outlined above. Because

$$cP^+ + cP^- = cP^+ + c(1 - P^+) = c \quad \forall c \in Cl(3)$$

and

$$Cl(3)P^+ \cap Cl(3)P^- = 0,$$

by the definition of direct sum we can identify $Cl(3)$ with $\mathbb{H} \times \mathbb{H}$.

3.3 Orbits of Equivalence and $Cl(4)$

Although $Cl(3)$'s solution is quite elegant, it would be incredibly ungainly to compute the matrices for every problem. Instead, we can describe them. One of the easiest approaches to this problem is to categorize the basis elements of $Cl(n)$ by the elements that the projection relates them to as we saw in Section 3.2. With that in mind, let us explore $Cl(4)$.

First, we note that γ_{123} does not lie in the center of $Cl(4)$, as

$$\gamma_4\gamma_{123} = -\gamma_{123}\gamma_4.$$

Because of this, our minimum representation does not necessarily contain a multiplicative identity as was the case with $Cl(3)$. Still, it resolves quite simply.

Example 3.3.1. Let $P^+ = \frac{1+\gamma_{123}}{2}$ as in Section 3.2. Then by (3.2.1), we can determine that inside of the subspace $Cl(4)P^+$, we have $1 \sim \gamma_{123}$. Together, they form the equivalence class $\{1, \gamma_{123}\}$. This will be referred to as the orbit of 1. Similarly, we compute as before

$$\gamma_1 P^+ = \gamma_1(1)P^+ = \gamma_1\gamma_{123}P^+ = -\gamma_{23}P^+,$$

creating the equivalence class $\{\gamma_1, -\gamma_{23}\}$. Using this methodology, we discover the following equivalence classes: $\{\gamma_2, \gamma_{13}\}$, $\{\gamma_3, -\gamma_{12}\}$, $\{\gamma_4, -\gamma_{1234}\}$, $\{\gamma_{14}, \gamma_{234}\}$, $\{\gamma_{24}, -\gamma_{134}\}$ and $\{\gamma_{34}, \gamma_{124}\}$. By the same procedure, we can describe equivalence classes formed by the ideal of P^- , with $1 \sim -\gamma_{123}$ instead. \diamond

This representation fails to explore an important property of $Cl(4)$. In $Cl(4)$, γ_{123} is not the only involution in the canonical basis. We can also show that γ_{124} , γ_{134} and γ_{234} are also involutions. These elements are identical to γ_{123} under the appropriate permutation of indices on the Clifford generators.

Definition 3.3.2. Let ω_1 and ω_2 be involutions in $Cl(n)$. The representation constructed using the projection $\frac{1+\omega_1}{2}$ is equivalent to the representation constructed using $\frac{1+\omega_2}{2}$ as a projection if we can permute the indices of the various γ_i 's in $Cl(n)$ such that $\omega_1 = \omega_2$. \triangle

The following is an example of two equivalent representations.

Example 3.3.3. Let $\omega = \gamma_{123}$ inside of $Cl(4)$. If we were to permute the indices such that $1 \leftrightarrow 4$, then $\omega = \gamma_{423} = -\gamma_{243} = \gamma_{234}$. Then for $P_1^+ = \frac{2+\gamma_{123}}{2}$ and $P_2^+ = \frac{1+\gamma_{234}}{2}$, we have

$$Cl(4)P_1^+ = Cl(4)P_2^+.$$

\diamond

However, not all involutions are equivalent. For example.

$$\gamma_{1234}\gamma_{1234} = \gamma_{123}\gamma_4\gamma_{1234} = \gamma_{123}(-\gamma_{123})(\gamma_4\gamma_4) = (-1)(-1) = 1. \quad (3.3.1)$$

No matter how we permute the indices of Clifford generators in γ_{123} , we will never construct γ_{1234} . This distinction demands terminology.

Definition 3.3.4. Let a be a canonical basis element of $Cl(n)$. We say that a has *weight* m , or $|a| = m$, where m is the number of Clifford generators in a . \triangle

Having established that γ_{123} and γ_{1234} have different weights, we will now construct a representation of $Cl(4)$ using γ_{1234} in our projection element.

Example 3.3.5. Let $P^+ = \frac{1+\gamma_{1234}}{2}$. By factoring γ_{1234} out of P^+ , we have the relation $1 \sim \gamma_{1234}$. Using this, we then compute the following equivalence classes: $\{1, \gamma_{1234}\}$, $\{\gamma_1, -\gamma_{234}\}$, $\{\gamma_2, \gamma_{134}\}$, $\{\gamma_3, -\gamma_{124}\}$, $\{\gamma_4, \gamma_{123}\}$, $\{\gamma_{12}, -\gamma_{34}\}$, $\{\gamma_{13}, \gamma_{24}\}$ and $\{\gamma_{14}, -\gamma_{23}\}$. These, along with the corresponding equivalence classes of P^- , can be used to describe any element of $Cl(4)$. Notice that each equivalence class has one element containing γ_4 and one without it. If we choose to look only at the elements that do not contain γ_4 , we can construct a correspondence between the equivalence classes of $Cl(4)$ and the basis elements of $Cl(3)$.

With some manipulation, the same could be shown of Example 3.3.1. However, because P^+ does not commute with γ_1 , we cannot cancel multiple instances of the projection for $Cl(4)P^+$ as we could for $Cl(3)P^+$, so P^+ is not an identity element in $Cl(4)P^+$. \diamond

It turns out that there are other involutions outside of the canonical basis. For example,

$$\begin{aligned} (\gamma_{123} + \gamma_4 + \gamma_{1234})^2 &= 1 + \gamma_{1234} + \gamma_4 - \gamma_{1234} - 1 + \gamma_{123} - \gamma_4 - \gamma_{123} + 1 \\ &= 1 - 1 + 1 + \gamma_{123} - \gamma_{123} + \gamma_4 - \gamma_4 + \gamma_{1234} - \gamma_{1234} = 1. \end{aligned}$$

We have chosen to restrict ourselves exclusively to involutions from the canonical basis.

However, in splitting $Cl(4)$, we arrive at a vector space quite similar to $Cl(3)$. Can we split this vector space using multiple projection elements?

Example 3.3.6. Let $P_1^+ = \frac{1+\gamma_{1234}}{2}$ and $P_2^+ = \frac{1+\gamma_{123}}{2}$. Inside of $Cl(4)P_1^+$, we compute

$$\frac{1 + \gamma_{123}}{2} = \frac{1 + \gamma_4}{2}$$

as $\gamma_4 P_1^+ = \gamma_{123} P_1^+$. Because $\gamma_4^2 + 1 = 0$, it has the eigenvectors i and $-i$, and so is not diagonalizable over \mathbb{R} . Then $\frac{1+\gamma_4}{2}$ is not diagonalizable and therefore is not a projection.

Therefore it fails to project onto $Cl(4)P_1^+$. Alternatively, we can show that $Cl(4)P_2^+ P_1^+$ fails to differ from $Cl(4)P_2^- P_1^+$ as follows:

$$\begin{aligned} P_2^+ P_1^+ &= \frac{1 + \gamma_{123}}{2} \frac{1 + \gamma_{1234}}{2} = -(\gamma_4)^2 \frac{1 + \gamma_4}{2} \frac{1 + \gamma_{1234}}{2} \\ &= \gamma_4 \frac{1 - \gamma_4}{2} \frac{1 + \gamma_{1234}}{2} \\ &= \gamma_4 \frac{1 - \gamma_{123}}{2} \frac{1 + \gamma_{1234}}{2} = \gamma_4 P_2^- P_1^+. \end{aligned}$$

The element P_2^+ ceases to be a projection in $Cl(4)P_1^+$, as it can be turned into $1 - P_2^+ = P_2^-$, albeit with a factor of γ_4 . The presence of P_1^+ on the right prevents P_2^+ from functioning as a projection. Were we to switch the two, a similar situation would arise. \diamond

From this example, we know $Cl(4)$ can be split only once using projections constructed from involutions in its canonical basis. However, this will not always be the case for $Cl(n)$.

3.4 Multiple Projection Elements

So far, we have been unable to use multiple projection elements at once. The inability to use the distinct projection elements in $Cl(4)$ is not universal to $Cl(n)$. The following criteria tells us when we can combine projection elements.

First, we introduce the following terminology:

Definition 3.4.1. Two matrices $A, B \in End(V)$ are *simultaneously diagonalizable* if there exists an invertible matrix $Q \in End(V)$ such that both QAQ^{-1} and QBQ^{-1} are diagonal matrices. \triangle

Then, we introduce an important result about simultaneously diagonalizable matrices.

Lemma 3.4.2. *Let $A, B \in End(V)$. Then A and B are simultaneously diagonalizable if and only if A and B commute.*

The proof of this lemma is omitted, but is given as an exercise in [4].

Lemma 3.4.3. *For involutions ω_1 and ω_2 , with the corresponding projections $P_1^+ = \frac{1+\omega_1}{2}$ and $P_2^+ = \frac{1+\omega_2}{2}$, if*

$$P_1^+ P_2^+ = P_2^+ P_1^+$$

then $P_1^+ P_2^+$ is a projection.

Proof. \Rightarrow Assume

$$P_1^+ P_2^+ = P_2^+ P_1^+.$$

Then

$$\begin{aligned} (P_1^+ P_2^+)^2 &= P_1^+ P_2^+ P_1^+ P_2^+ \\ &= P_1^+ P_1^+ P_2^+ P_2^+ = P_1^+ P_2^+, \end{aligned}$$

and by Lemma 2.3.5 is a projection. \square

The converse can be shown given the constraints of our problem.

Lemma 3.4.4. *Let ω_1 and ω_2 be involutions in $Cl(n)$, with the corresponding projections $P_1^+ = \frac{1+\omega_1}{2}$ and $P_2^+ = \frac{1+\omega_2}{2}$, such that $\omega_1\omega_2 = -\omega_2\omega_1$. Then $Cl(n)P_2^+P_1^+ = Cl(n)P_1^+$.*

Proof. By the reasoning in 3.2.1, we have $(1)P_1^+ = \omega_1P_1^+$. Then in $Cl(n)P_1^+$, we have $\omega_2 \sim \omega_2\omega_1$. We compute

$$\begin{aligned} (\omega_2\omega_1) &= \omega_2\omega_1\omega_2\omega_1 \\ &= -\omega_2\omega_2\omega_1\omega_1 \\ &= (-1)(1)(1) = -1 \end{aligned} \tag{3.4.1}$$

so $\omega_2\omega_1$ is not an involution. Then, by the reasoning found in Example 3.3.6, we see that $Cl(n)P_2^+P_1^+ = Cl(n)P_1^+$.

□

Assume we have two commuting projections P_1 and P_2 in $Cl(n)$. We still don't know P_2 , will act on subspaces created by P_1 . However, the second projection continues to meet most of the criterion necessary to satisfy Lemma 3.1.1. Therefore it comes as no surprise that P_2 splits $Cl(n)P_1$.

Lemma 3.4.5. *Let ω_1 and ω_2 be orthogonal commuting involutions in the canonical basis of $Cl(n)$ that are also orthogonal to 1. Additionally, let $P_1^+ = \frac{1+\omega_1}{2}$ and $P_2^+ = \frac{1+\omega_2}{2}$ be projections. Then P_2^+ splits the vector space $Cl(n)P_1^+$ into two subspaces of equal dimension.*

Proof. We know ω_1 and ω_2 are diagonalizable because all projections are diagonalizable and the identity matrix is always diagonal, as was shown in Lemma 3.1.1. By hypothesis, they commute and so are simultaneously diagonalizable by Lemma 3.4.2. Then ω_2 and P_1^+ are simultaneously diagonalizable as well.

Now, by the reasoning found in proving Lemma 3.1.1, we see that the eigenvalues of l_ω are 1 and -1 and its trace is zero, so we can construct the projection element P_2^+ inside of $Cl(n)P_1^+$. □

Because ω and $-\omega$ are both involutions, the above Lemma holds for $P_1^+ P_2^-$, $P_1^- P_2^+$ and $P_1^- P_2^-$.

Now we need to find which involutions commute. Because we have limited ourselves to involutions from the canonical basis, we can enormously simplify this problem.

Lemma 3.4.6. *Let $a, b \in Cl(n)$ such that $a = \gamma_{i_1} \cdots \gamma_{i_p}$ and $b = \gamma_{j_1} \cdots \gamma_{j_m}$ (i.e. a and b are canonical basis elements with weights p and m respectively). Then a and b commute if and only if*

$$|\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_m\}| = mp \pmod{2}.$$

Proof. Let $\gamma_{i_k} \cap \{\gamma_{j_1}, \dots, \gamma_{j_m}\} = \emptyset$. Then $\gamma_{i_k} b = (-1)^m b \gamma_{i_k}$ as γ_{i_k} switches places a total of m times. Let $\gamma_{i_l} \in \{\gamma_{j_1}, \dots, \gamma_{j_m}\}$. Then for $\gamma_{i_l} = \gamma_{j_k}$,

$$\begin{aligned} \gamma_{i_l} b &= \gamma_{j_k} \gamma_{j_1} \cdots \gamma_{j_{k-1}} \gamma_{j_k} \gamma_{j_{k+1}} \cdots \gamma_{j_m} \\ &= (-1)^{k-1} \gamma_{j_1} \cdots \gamma_{j_{k-1}} \gamma_{j_k} \gamma_{j_k} \gamma_{j_{k+1}} \cdots \gamma_{j_m} \\ &= (-1)^{k-1} (-1)^{m-k} \gamma_{j_1} \cdots \gamma_{j_{k-1}} \gamma_{j_k} \gamma_{j_{k+1}} \cdots \gamma_{j_m} \gamma_{j_k} = (-1)^{m-1} b \gamma_{i_l} \end{aligned}$$

Then for $c = |\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_m\}|$,

$$ab = (-1)^{c(m-1)} (-1)^{(p-c)m} ba = (-1)^{pm-c} ba.$$

Therefore, if $c = pm \pmod{2}$, then $ab = ba$ and if $c \neq pm \pmod{2}$, then $ab = -ba$. \square

At this juncture, we identify all of the involutions in the canonical basis of $Cl(n)$.

Lemma 3.4.7. *Let a be in the canonical basis of $Cl(n)$ with weight m . Then a is an involution if and only if*

$$m = 3 \pmod{4} \quad \text{or} \quad m = 0 \pmod{4}.$$

Proof. Let $m = 0 \pmod{4}$ and $m = 4k$. Then we can break a down into k blocks of the form $\sigma = \gamma_{efgh}$ where $e, f, g, h \in \{1, \dots, n\}$ with no intersections. Such blocks commute

by Lemma 3.4.6 as they share no Clifford generators and are both of even weight. By equation (3.3.1),

$$(\omega_i)^2 = 1.$$

Let σ_i be the i th such block. Then $a = \sigma_1 \cdot \sigma_k$ and

$$\begin{aligned} a^2 &= \sigma_1 \cdots \sigma_k \sigma_1 \cdots \sigma_k \\ &= \sigma_1 \sigma_1 \cdots \sigma_k \sigma_k = 1^k = 1. \end{aligned}$$

Now let $m = p \pmod 4$ where $0 < p < 4$. Then $m = 4k + p$ and using the above notation,

$a = \sigma_1 \cdots \sigma_k \gamma_{j_1} \cdots \gamma_{j_p}$ where γ_{j_i} is a Clifford generator. Then

$$\begin{aligned} a^2 &= \sigma_1 \cdots \sigma_k \gamma_{j_1} \cdots \gamma_{j_p} \sigma_1 \cdots \sigma_k \gamma_{j_1} \cdots \gamma_{j_p} \\ &= \sigma_1 \cdots \sigma_k \sigma_1 \cdots \sigma_k \gamma_{j_1} \cdots \gamma_{j_p} \gamma_{j_1} \cdots \gamma_{j_p} \\ &= \gamma_{j_1} \cdots \gamma_{j_p} \gamma_{j_1} \cdots \gamma_{j_p} = (\gamma_{j_1} \cdots \gamma_{j_p})^2 \end{aligned}$$

using Lemma 3.4.6. We have already shown that if $p = 1, 2$, then $(\gamma_{j_1} \cdots \gamma_{j_p})^2 = -1$ and if $p = 3$, then $(\gamma_{j_1} \cdots \gamma_{j_p})^2 = 1$. Thus a is an involution if and only if $m = 0, 3 \pmod 4$. \square

With the above tools outlined, we can easily represent various $Cl(n)$ minimally.

3.5 Examples

Using the lemmas from Section 3.4, we can now represent various Clifford algebras minimally. Some issues of notation will arise, most notably that the orbits of 1 grow exponentially. Because the equivalence class of 1 defines all other equivalence classes, we need only to describe it to characterize our entire reduced representation. Let's begin.

Example 3.5.1. In $Cl(5)$, let $P_1^+ = \frac{1+\gamma_{123}}{2}$ and $P_2^+ = \frac{1+\gamma_{145}}{2}$, which commute by Lemma 3.4.6. From Examples 3.3.1 and 3.3.5, we have $1P_1^+ = \gamma_{123}P_1^+$ and $1P_2^+ = \gamma_{145}P_2^+$. Therefore,

$$\begin{aligned} 1P_1^+P_2^+ &= \gamma_{123}P_1^+P_2^+ = \gamma_{123}(1)P_2^+P_1^+ \\ &= \gamma_{123}\gamma_{145}P_2^+P_1^+ = -\gamma_{2345}P_1^+P_2^+. \end{aligned}$$

From this, we determine that the equivalence class of 1 inside of $Cl(5)P_1^+P_2^+$ is $\{1, \gamma_{123}, \gamma_{145}, -\gamma_{2345}\}$. By similar reasoning, the equivalence class of 1 is $\{1, \gamma_{123}, -\gamma_{145}, \gamma_{2345}\}$ in $Cl(5)P_1^+P_2^-$, $\{1, -\gamma_{123}, \gamma_{145}, \gamma_{2345}\}$ in $Cl(5)P_1^-P_2^+$ and $\{1, -\gamma_{123}, -\gamma_{145}, -\gamma_{2345}\}$ in $Cl(5)P_1^-P_2^-$. As the initial representation is thirty-two dimensional, each of these subspaces is eight dimensional by Lemmas 3.1.1 and 3.4.5. Then $Cl(5)$ can be represented using four 8×8 matrices rather than a single 32×32 matrix. Our minimal representation contains one fourth as many entries as the standard matrix. We could just as easily let γ_{2345} be one of our involutions and get identical results. \diamond

We could have continued on to find the remaining seven equivalence classes of $Cl(5)$ for each possible split (we know that there are eight total because there are 2^5 basis elements split into sets of four), but we have simplified this process. Still, our simplification is incomplete. In $Cl(5)$, we had to select from three projection elements but only used two. We must designate a system for choosing the involutions used to construct our projections.

Notice that the product of two commuting elements in $Cl(n)$ with odd weight will have an even weight. Then for any representation using multiple odd weight involutions in its projections, we can change all but one of them to even weight involutions. We will attempt to order our projections first by choosing as many even weight involutions as possible, then adding an odd one if necessary.

Example 3.5.2. By the above criteria, let $P_1^+ = \frac{1+\gamma_{1234}}{2}$ and $P_2^+ = \frac{1+\gamma_{125}}{2}$ in $Cl(5)$. Then the orbit of 1 in $Cl(5)P_1^+P_2^+$ is $\{1, \gamma_{1234}, \gamma_{125}, -\gamma_{345}\}$. Note how we use γ_1 and γ_2 in our set of generating involutions. The use of γ_1 and γ_2 is to highlight that these projections are distinct from one another. \diamond

This issue becomes more apparent in $Cl(6)$, where there are eight involutions in the equivalence class of 1 from which to choose.

Example 3.5.3. In $Cl(6)$, let $P_1^+ = \frac{1+\gamma_{1234}}{2}$, $P_2^+ = \frac{1+\gamma_{1256}}{2}$ and $P_3^+ = \frac{1+\gamma_{135}}{2}$. Then we can combine the basis element components of the P_i in every possible combination. We now compute the equivalence class of 1 as $\{1, \gamma_{1234}, \gamma_{1256}, -\gamma_{3456}, \gamma_{135}, \gamma_{245}, -\gamma_{236}, \gamma_{146}\}$. Note that this order was determined by finding all elements in the equivalence class of 1 with the first two projection elements before incorporating the third.

As we have three projections which are either positive or negative, there are $2^3 = 8$ possible combinations of positive and negative projections, so this representation splits the sixty four dimensional $Cl(6)$ into eight subspaces, each of which can be represented by an 8×8 matrix. \diamond

The inclusion of all eight elements in the equivalence class of 1 is unnecessary, as they are all determined by the non-one components of the projection elements. Instead, we can define this representation entirely by the involutions of each projection. Henceforth, we will refer to this representation as the representation generated by γ_{1234} , γ_{1256} and γ_{135} . Because the third involution has an odd weight, there is no way it can be the product of the first two involutions.

As the representations grow more and more unwieldy to describe, we need terminology to describe them succinctly. We will refer to the various representations of $Cl(n)$ with m distinct representations as $R(n)_i$ where $i \in \{1, \dots, m\}$. Although representations can be described by referring to orbits of 1 or the choice of projection elements, we choose to think of elements in $R(n)$ as the involutions found in any choice of projections elements, although we will use the standard choices whenever possible. The sign of these involutions will always be written as positive but can be negative as well.

Let $R(a)$ contain the involutions ω_1 , ω_2 and ω_3 while $R(b)$ contains only ω_1 and ω_2 . Then we say $R(b) \in R(a)$ and $R(a) = \{R(b), \omega_3\}$.

Example 3.5.4. By definition $R(5) = \{\gamma_{1234}, \gamma_{125}\}$ and $R(6) = \{\gamma_{1234}, \gamma_{3456}, \gamma_{135}\}$. If we permute the indices in $R(6)$ such that $2 \leftrightarrow 3$, then first two involutions in $R(6)$ become

$$\{\gamma_{1324}, \gamma_{125}\} = \{-\gamma_{1234}, \gamma_{125}\}$$

However, because the sign of these involutions can be ignored, we have

$$\{\gamma_{1234}, \gamma_{125}\} = R(5) \in R(6)$$

after permuting indices in $R(6)$. Additionally, we can say $R(6) = \{R(5), \gamma_{2456}\}$, although this is not the standard notation. \diamond

We now have developed all of the techniques and vocabulary necessary to define $Cl(7)$ and $Cl(8)$. Let's begin.

Example 3.5.5. In $Cl(7)$, we take the initial generating set of $\{\gamma_{1234}, \gamma_{1256}, \gamma_{1357}, \gamma_{127}\}$.

We compute

$$\gamma_{1234}\gamma_{1256} = \gamma_{1212}\gamma_{3456} = -\gamma_{3456}$$

so γ_{1357} is not their product. In general, because it does not pair γ_1 and γ_2 , it must be distinct. Then the above set of involutions generates our representation of $Cl(7)$, which we call $R(7)$. As $R(7)$ is generated by four projections, it can be represented by sixteen 8×8 matrices. \diamond

When an additional Clifford generator is added to $Cl(7)$, making $Cl(8)$, there are no new projection elements that commute with those found in the representation of $Cl(7)$, so $R(7)$ is a minimal representation of $Cl(8)$. In fact, were we to create an endless supply of Clifford generators, the only involutions that would commute with the representation $R(7)$ are of the form ab where $a \in R(7)$ and b is an even weight involution where if $\gamma_i \in b$, then $i > 7$. However, that does not mean the addition of γ_8 allows for no new representations.

Example 3.5.6. In $Cl(8)$, we take the initial generating set of $\{\gamma_{1234}, \gamma_{1256}, \gamma_{1278}, \gamma_{1357}\}$.

The first three involutions each contain a unique pair of Clifford generators and the fourth

contains γ_1 but does not have γ_2 . Therefore they are multiplicatively distinct and can be used to generate a representation of $Cl(8)$, which we call $R(8)$.

Because four projections are used to construct $R(8)$, just as in $R(7)$, we can split $Cl(8)$ into sixteen subspaces. Each of these subspaces will be sixteen dimensional, so $Cl(8)$ can be represented with sixteen 16×16 matrices rather than a single 256×256 matrix used in the standard representation. \diamond

Similarly to $R(7)$, the only involutions containing Clifford generators from $R(8)$ are products of elements in $R(8)$ and involutions outside of $Cl(8)$. However, because every involution in this representation has even weight, outside involutions can be either even or odd. Using this fact, along with a bit of additional machinery, we can construct a minimal representation for the general $Cl(n)$.

Lemma 3.5.7. *For every representation ρ of $Cl(n)$ that is not minimal, there exists another projection σ that commutes with ρ such that $\sigma\rho$ is a minimal projection.*

A proof of this lemma can be found in [1, Dimakis].

Definition 3.5.8. Let A and B be representations of $Cl(a)$ and $Cl(b)$ respectively. Then we define AB to be the representation $A \times B$ in $Cl(a + b)$ where A is expressed entirely in terms of the first a Clifford generators and B is expressed entirely in terms of the last b Clifford generators. \triangle

The following example outlines an important advantage of this notation.

Example 3.5.9. Let $\omega_1 = \gamma_{1234}$ and $\omega_2 = \gamma_{5678}$ in $Cl(8)$. Then each such ω generates the representation $R(4)$ so the two combined generate the representation $R(4)R(4) = R(4)^2$. \diamond

We are now prepared to demonstrate a minimal representation of $Cl(n)$.

Example 3.5.10. We have already found minimal representations of $Cl(n)$ for $n \leq 8$. Let $n > 8$. Then $n = 8k + c$ for some integers k and $c < 8$. We construct the representation

$$R(8)^k R(c)$$

where each $R(8)$ is a representation on eight distinct Clifford generators inside of $Cl(n)$. Because $R(8)$ is a minimal representation of $Cl(8)$ that restricts new projections from containing Clifford generators inside of it, no new projection elements can be added to this representation. Therefore, by Lemma 3.5.7 it is minimal.

Each instance of $R(8)$ adds 4 projection elements to our representation. Let d be the number of projections necessary to construct a minimal representation of $Cl(c)$. Then the minimal representation of $Cl(n)$ breaks it down into 2^{4k+d} subspaces with dimension $2^{n-(4k+d)}$. \diamond

Were we to perform a similar process to the one above, only using a single $R(7)$, we would be restricted entirely to even weight involutions in constructing our projection elements. As we would then have one additional Clifford generator to work with, showing that if a minimal representation of $Cl(n)$ contains c projections, then we can find c commuting projections with even weight in $Cl(n+1)$.

Finding the minimal representation of $Cl(n)$ has proven relatively easy. However, the above procedure is only one construction. We have just seen that there are two distinct ways of minimally representing $Cl(8)$, as many as there were for $Cl(4)$. How many ways will there be to represent $Cl(9)$? And $Cl(10)$? What of $Cl(n)$? Computing these representations is a good first step, but the sheer size and introduction of new projecting elements of length seven and eight, eleven and twelve will make brute force computations impossible, especially in the general case. Now is the time to turn towards observation, theory and abstraction, hopefully gaining a sharp intuition from these several cases.

4

Patterns in Representations

4.1 Representations: $Cl(9) - Cl(16)$

In order to find a formula for the number of minimal representations of $Cl(n)$, we should have an intuition for representing $Cl(n)$ where n is some large number. In order to acquire such intuition, we should represent a great many Clifford algebras in as many ways as possible. Once we have some data in the form of representations and an awareness of how to work with it, we can begin searching for patterns. We will find all of the minimal representations of $Cl(9)$ through $Cl(16)$.

Before finding more minimal representations, we must deal with notational difficulties found in Clifford algebras with 10 or more Clifford generators. In order to maintain our simplified form for writing the products of Clifford generators, we define the following: $\gamma_{10} = \gamma_0$, $\gamma_{11} = \gamma_a$, $\gamma_{12} = \gamma_b$, $\gamma_{13} = \gamma_c$, $\gamma_{14} = \gamma_d$, $\gamma_{15} = \gamma_e$, and $\gamma_{16} = \gamma_f$. Now, we are prepared to explore these new, fertile grounds.

First, we need to develop terminology for describing a basic pattern found in many representations.

Definition 4.1.1. We define D_n to be the representation of $Cl(n)$ whose orbit of 1 is multiplicatively generated by the involutions $\{\gamma_{1234}, \gamma_{1256}, \dots, \gamma_{12(n-1)n}\}$ where n is even. Note that D_n is not necessarily minimal; in fact, it is only minimal when $n = 4$. \triangle

Inside of D -type representations, the Clifford generator γ_{2i-1} is always paired with γ_{2i} . For example, γ_1 always appears with γ_2 and γ_a always appears with γ_b .

Example 4.1.2. Using D_n notation, we can describe past representations. For example, $R(5)$ is the set of involutions that is multiplicatively generated by the set $\{D_4, \gamma_{125}\}$ as $D_4 = \{\gamma_{1234}\}$. We then say that $R(5) = \{D_4, \gamma_{125}\}$. Similarly, we have $R(8) = \{D_8, \gamma_{1357}\}$. \diamond

We are now prepared to present the representations of $Cl(9) - Cl(16)$, which we believe to be complete.

Example 4.1.3. In addition to describing all of the representations we have found, we note that representations of $Cl(n)$ contain the representations of $Cl(n - 1)$.

1. The following are minimal representations of $Cl(9)$:

$$R(7), R(8), R(9) = \{D_8, \gamma_{129}\}. \quad (4.1.1)$$

2. The following are minimal representations of $Cl(10)$:

$$R(7), R(8), R(9), R(10) = D_{10}. \quad (4.1.2)$$

3. The following are minimal representations of $Cl(11)$:

$$\begin{aligned} R(11)_1 &= \{R(7), \gamma_{890a}\}, \\ R(11)_2 &= \{R(8), \gamma_{90a}\}, \\ R(11)_3 &= \{D_{10}, \gamma_{12a}\}. \end{aligned} \quad (4.1.3)$$

Note that $R(9), R(10) \in R(11)_3$.

4. The following are minimal representations of $Cl(12)$:

$$R(11)_1, R(11)_2, R(11)_3,$$

and

$$\begin{aligned} R(12)_1 &= \{R(8), \gamma_{90ab}\}, \\ R(12)_2 &= D_{12}. \end{aligned} \tag{4.1.4}$$

5. The following are minimal representations of $Cl(13)$:

$$\begin{aligned} R(13)_1 &= \{R(7), \gamma_{890a}\gamma_{89bc}\}, \\ R(13)_2 &= \{R(8), \gamma_{90ab}\gamma_{90c}\}, \\ R(13)_3 &= \{D_{12}, \gamma_{12c}\}, \\ R(13)_4 &= \{D_{12}, \gamma_{13579ac}\}. \end{aligned} \tag{4.1.5}$$

Note that $R(11)_1 \in R(13)_1$, $R(11)_2, R(12)_1 \in R(13)_2$ and $R(11)_3 \in R(13)_3$.

6. The following are minimal representations of $Cl(14)$:

$$\begin{aligned} R(14)_1 &= \{R(7), \gamma_{890a}, \gamma_{89bc}, \gamma_{80bd}\}, \\ R(14)_2 &= \{R(8), \gamma_{90ab}, \gamma_{90cd}, \gamma_{9ac}\}, \\ R(14)_3 &= \{D_{12}, \gamma_{12c}, \gamma_{13579acd}\}, \\ R(14)_4 &= \{D_{14}, \gamma_{13579ac}\}. \end{aligned} \tag{4.1.6}$$

Note that $R(13)_1 \in R(14)_1$, $R(13)_2 \in R(14)_2$, $R(13)_3, R(13)_4 \in R(14)_3$ and $R(13)_4 \in R(14)_4$.

7. The following are minimal representations of $Cl(15)$:

$$\begin{aligned} R(15)_1 &= \{R(7), \gamma_{890a}, \gamma_{89bc}, \gamma_{80bd}, \gamma_{89de}\}, \\ R(15)_2 &= \{D_{14}, \gamma_{12e}, \gamma_{13579ace}\}. \end{aligned} \tag{4.1.7}$$

Note $R(14)_1, R(14)_2 \in R(15)_1$ and $R(14)_3, R(14)_4 \in R(15)_2$.

8. The following are minimal representations of $Cl(16)$:

$$R(15)_1, R(15)_2,$$

and

$$R(16)_1 = R(8)^2 \tag{4.1.8}$$

$$R(16)_2 = \{D_{16}, \gamma_{13579ace}\}.$$

◇

These representations will be used as the chief supporting evidence for any conjectures to be made in the remainder of the paper.

4.2 A New Notion of Equivalence

Using Example 4.1.3 as evidence, we outline the major conjecture of this project.

Conjecture 4.2.1. *Let ρ and σ refer to orbits of 1 that define representations of $Cl(n)$.*

If there exists a function $f : \rho \rightarrow \sigma$ that is bijective and weight preserving (for all $a \in \rho$,

$|f(a)| = |a|$), then ρ and σ are equivalent.

There are several clear benefits to applying this conjecture, rather than the definition of equivalence between projections. The computations necessary to establish equivalence between representations are much simpler as they can be applied across many representations without finding each permutation. Any confusion caused by the signs of individual projection elements can be ignored. Now we explore why this conjecture is likely to be true.

First of all, the converse is true. If there are more elements in one orbit of 1 with a given weight than in another, there is no way the two projections corresponding to these orbits are equivalent. No permutation can overcome a difference in weight. Additionally, it has proven a useful test for the examples in the previous chapter.

There are several approaches available to proving this conjecture. The first and more promising is to attempt an inductive step. Clearly, the standard representation is always equivalent to itself. Then assuming we have two equivalent representations, we add a projection element to each representation that preserves the weight correspondence between the two representations. Then we attempt to prove the new representations are also equivalent. This inductive step is terribly messy, splitting into many cases as there are multiple projection element types that could be added to a projection. Any proof using this methodology would rely on finding structural relations that limit the possible projections to be added.

The following might be the form one such case might take.

Conjecture 4.2.2. *Let ρ be the orbit of 1 defining a representation of $Cl(n)$. Let ω_1 and ω_2 be involutions of weight 4 that contain precisely two Clifford generators not found in involution within ρ but which commute with ρ . Then*

$$\left\{ \rho, \frac{1 + \omega_1}{2} \right\} = \left\{ \rho, \frac{1 + \omega_2}{2} \right\}.$$

Assuming this conjecture is true, we then have several promising leads to developing a formula for counting the number of distinct minimal representations of $Cl(n)$.

One important matter we have neglected to mention is data. We have representations all the way up to $Cl(16)$ from which we can compute the weights. Why haven't we included this data? The answer is simple. It does not support the conjecture.

Example 4.2.3. In the following table, the boldfaced values are the potential weights of involutions while each entry represents the number of involutions in a given representation with the weight attached to its column.

	0	3	4	7	8	11	12	15	16
$R(9)$	1	4	6	4	1				
$R(10)$	1		10		5				
$R(11)_1$	1	7	8	8	7	1			
$R(11)_2$	1	1	14	14	1	1			
$R(11)_3$	1	5	10	10	5	1			
$R(12)_1$	1		15		15		1		
$R(12)_2$	1		15		15		1		
$R(13)_1$	1	7	10	22	21	3			
$R(13)_2$	1	2	15	28	15	2	1		
$R(13)_3$	1	6	15	20	15	6	1		
$R(13)_4$	1		15	32	15		1		
$R(14)_1$	1	7	14	50	49	7			
$R(14)_2$	1	4	17	57	42	4	3		
$R(14)_3$	1		21	64	35		7		
$R(14)_4$	1	6	15	52	47	6	1		
$R(15)_1$	1	7	21	99	99	21	7	1	
$R(15)_2$	1	7	21	99	99	21	7	1	
$R(16)_1$	1		28		198		28		1
$R(16)_2$	1		28		198		28		1

◇

Now we are confronted with several counterexamples. The conjecture proves to be wrong in three separate cases:

1. $R(12)_1$ and $R(12)_2$
2. $R(15)_1$ and $R(15)_2$
3. $R(16)_1$ and $R(16)_2$

Perhaps by exploring the relationships between these representations, we can find new avenues to explore. Were we to find a way to classify the counterexamples to our conjecture, we could salvage the computational advantage it bestows.

In examining each of the three counterexamples, one of the representations contains $R(8)$ while the other is built from a D type representation. However, $R(14)_2$ and $R(14)_4$ also fits this description as do $R(11)_2$ and $R(11)_3$. Each contains an element with weight n , but so do $R(11)_2$ and $R(11)_3$. The only remaining avenue worth exploring is whether

this relationship relies on the fact that all three counterexamples contain the only representations using all n elements. We find this possibility highly improbable, especially as the number of possible projections grows dramatically with the increase of n .

4.3 Towards Counting Representations

Recalling our argument for the dimension of $Cl(n)$, we can view each element of the canonical basis as either containing or failing to contain γ_i . Then there is an injective map between the canonical basis of $Cl(n)$ and the elements of $P(S)$, the power set of S where $|S| = n$. As we have seen so far, the signs found in our representations have proven quite mutable. We suspect that sign will prove negligible in counting the number of distinct representations for $Cl(n)$. Then, by developing the relationship between the canonical basis of $Cl(n)$ and $P(S)$, we gain a new area for computing the number of possible representations.

Definition 4.3.1. Let A and B be sets. Then the *symmetric difference* of A and B is

$$A\Delta B = (A \cup B) - (A \cap B).$$

△

Then we have the following:

Lemma 4.3.2. Let S be the set $\{1, \dots, n\}$. Then $(P(S), \Delta)$ is isomorphic to \mathbb{Z}_2^n .

Proof. Let $a \in \mathbb{Z}_2^n$. Then a is of the form

$$a = (a_1, a_2, \dots, a_n)$$

where $a_i = 0$ or 1 . Let $T_a \subset S$ be the set of all j such that $a_j = 1$. We compute

$$a + b = (a_1, \dots, a_n) + ((b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n),$$

where $(a_i + b_i) = 1$ if $a_i \neq b_i$ and 0 otherwise. Additionally, we compute $T_a \Delta T_b = T_a \cup T_b - T_a \cap T_b$. Then $i \in T_{a+b}$ if $a_i \neq b_i$ and $i \notin T_{a+b}$ if $a_i = b_i$. Therefore, the operations are equivalent over their respective groups.

□

An important observation: this group is also isomorphic to multiplication of the canonical basis elements of $Cl(n)$ ignoring sign.

By expanding the number of ways we can look at Clifford algebras, we gain new insights that may lead to a formula for the number of distinct minimal representations of $Cl(n)$.

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