STRUCTURE OF THE PEAK HOPF ALGEBRA OF QUASISYMMETRIC FUNCTIONS

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ABSTRACT. We analyze the structure of Stembridge's peak algebra, showing it to be a free commutative algebra (specifically a shuffle algebra) over $\mathbb{Q}$, a cofree graded coalgebra, and a free module over Schur's Q-function algebra. Our analysis builds on combinatorial properties of a new monomial-like basis for the peak algebra. We describe the product, coproduct, and antipode on this basis. The dual of the monomial basis corresponds to a family of flag-enumeration functionals on Eulerian posets. We show that nonnegativity of these functionals on all Gorenstein* posets is equivalent to a special case of the combinatorial Hopf conjecture of Charney and Davis. We describe a new basis of eigenvectors for Stembridge's descents-to-peaks map that relates triangularly to the monomial basis.

1. INTRODUCTION

The peak algebra Π first arose in Stembridge's theory of enriched P-partitions [19] as a setting in which to generalize the combinatorial study of Schur's Q-functions. In earlier work, Gessel [9] had detailed a similar interplay between the algebra Qsym of quasisymmetric functions, the theory of (ordinary) P-partitions, and (ordinary) Schur functions. In light of the analogy between the ordinary and enriched theories, this paper draws on established notions of Qsym to analyze the Hopf algebra structure of Π. Most of our results can be regarded as peak analogues of well-known facts about quasisymmetric functions.

Our analysis of Π begins in Section 2 with the introduction of a basis consisting of *monomial peak functions.* We show that Stembridge's descents-to-peaks map $\Theta : \mathsf{Qsym} \to \Pi$ takes ordinary monomial quasisymmetric functions to monomial peak functions (up to sign). In

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analogy with Qsym, we describe the product and coproduct of monomial peak functions in terms of quasi-shuffles and concatenation, respectively. We also give a simple formula for the antipode on this basis.

In Section 3 we study the restriction of Θ to each homogeneous graded component of Π. Billera et al. [4] diagonalized this map and interpreted it as a random walk on peak sets. Here we describe a new basis of eigenvectors that relates triangularly to the monomial peak functions. Our basis is an important tool in determining the algebraic structure of Π.

The main goal of Section 4 is to formulate and prove peak analogues of two results by Malvenuto and Reutenauer [11, 12] on the structure of Qsym. We show that the dual Hopf algebra of Π is a concatenation Hopf algebra (Theorem 4.1) and that Π is a free commutative algebra (specifically a shuffle algebra) and a free module over Schur's Q-function algebra (Corollary 4.2). Our proofs employ the eigenvectors from Section 3. Next we define a coalgebra grading on Π which gives it the structure of a cofree graded coalgebra.

Finally, in Section 5 we consider some connections between Π and the theory of flag-enumeration in Eulerian posets. Bergeron et al. [2] showed that the dual Hopf algebra of Π can be naturally identified with the Billera-Liu algebra of functionals on Eulerian posets [5]. From this viewpoint, each linear basis for Π is dual to a family of flag-enumerative functionals on Eulerian posets. The most common such family, known as the **cd**-index, was recently shown by Billera et al. [4] to be dual to Stembridge's basis of (ordinary) peak functions. Babson had earlier observed a connection between the **cd**-index and the combinatorial Hopf conjecture of Charney and Davis [6]. Building on these facts, we shall see that nonnegativity of the monomial peak functionals on all Gorenstein* posets is equivalent to an important special case of the Charney-Davis conjecture. Our work essentially reformulates some of Reading's results [13] on the Charney-Davis conjecture.

We gather some basic facts and definitions in the remainder of the Introduction. Throughout this paper, we use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{P} = \{1, 2, \ldots\}$, $[n] = \{1, 2, \ldots, n\}$, and $[n, m] = \{n, n+1, n+2, \ldots, m\}$ for $n, m \in \mathbb{N}$. In particular, $[0] = \emptyset$ and $[n, m] = \emptyset$ if $n > m$. If the elements of a set $\{a_1, a_2, \ldots, a_k\} \subset \mathbb{N}$ are listed so that $a_1 < a_2 < \cdots < a_k$, then we may denote this set by $\{a_1, a_2, \ldots, a_k\}_<$.

1.1. **Quasisymmetric functions.** Let Qsym denote the Hopf algebra of quasisymmetric functions in the variables $x_1, x_2, \ldots$ with coefficients in $\mathbb{Q}$. As a vector space, Qsym has a basis consisting of the *monomial* *functions*

$$M_\alpha = \sum_{i_1<\cdots<i_k} x_{i_1}^{\alpha_1}\cdots x_{i_k}^{\alpha_k} \in \mathbb{Q}[[x_1,x_2,\ldots]],$$

where $k \geq 0$ and $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a vector of positive integers, that is, a *composition* of $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_k$. The *length* of α is $l(\alpha)=k$, and each α_i is called a *part* of α. The case $k=0$ corresponds to $\alpha=\emptyset$, the unique composition of 0. We define $M_\emptyset = 1$.

Let $\mathrm{Comp}(n)$ denote the set of compositions of n. Consider the partial order on $\mathrm{Comp}(n)$ generated by the cover relations

$$(1.1)\qquad (\alpha_1,\ldots,\alpha_i+\alpha_{i+1},\ldots,\alpha_k) < (\alpha_1,\ldots,\alpha_i,\alpha_{i+1},\ldots,\alpha_k).$$

The resulting poset is isomorphic to the lattice of subsets of $[n-1]$ ordered by inclusion. The usual isomorphism associates $\alpha=(\alpha_1,\ldots,\alpha_k)$ to the set

$$S_\alpha = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\}.$$

In addition to the monomial functions, Qsym has another important basis consisting of the *fundamental quasisymmetric functions* F_α, defined for all $\alpha\in\mathrm{Comp}(n)$, $n\geq 0$, by the following equivalent sets of relations:

$$(1.2)\qquad F_\alpha = \sum_{\beta\in\mathrm{Comp}(n):\beta\geq\alpha} M_\beta,$$

$$(1.3)\qquad M_\alpha = \sum_{\beta\in\mathrm{Comp}(n):\beta\geq\alpha} (-1)^{l(\beta)-l(\alpha)} F_\beta.$$

Alternately, F_α can be defined as the *weight enumerator* for all P-partitions of a labeled chain of size $|\alpha|$ with *descent set* S_α; see [9, 17] for details on this approach.

Let Qsym_n denote the linear span of $\{M_\alpha \mid \alpha\in\mathrm{Comp}(n)\}$, so that $\dim_{\mathbb{Q}}\mathsf{Qsym}_n = 2^{n-1}$. It is easy to verify that $\mathsf{Qsym} = \bigoplus_{n\geq 0}\mathsf{Qsym}_n$ is a graded subalgebra of $\mathbb{Q}[[x_1,x_2,\ldots]]$. Multiplication of monomial functions is described in terms of *quasi-shuffles*. In order to define a quasi-shuffle, it will be convenient to represent a composition by a map $\alpha : X\to\mathbb{P}$ for some chain $X=\{x_1,\ldots,x_k\}$, so that $(\alpha(x_1),\ldots,\alpha(x_k))$ is a composition in the usual sense.

Definition 1.1. A *quasi-shuffle* of two compositions $\alpha : X\to\mathbb{P}$ and $\beta : Y\to\mathbb{P}$ is a pair (τ,γ), where τ is a strictly order-preserving map from the disjoint union $X\cup Y$ onto $[k]$ for some $k>0$, and $\gamma=(\gamma_1,\ldots,\gamma_k)$ is the composition given by

$$\gamma_i = \begin{cases} \alpha(x)+\beta(y), & \text{if } \tau(x)=\tau(y)=i, \\ \alpha(x), & \text{if } \tau(x)=i \text{ and } \tau(y)\neq i \text{ for all } y\in Y, \\ \beta(y), & \text{if } \tau(y)=i \text{ and } \tau(x)\neq i \text{ for all } x\in X. \end{cases}$$

The product of two monomial functions satisfies

$$M_\alpha M_\beta = \sum_{(\tau,\gamma)} M_\gamma, \tag{1.4}$$

where the sum is over all quasi-shuffles of α and β [7, Lemma 3.3]. For example,

$$M_{(1,2)}M_{(1)} = M_{(1,2,1)} + M_{(1,3)} + 2M_{(1,1,2)} + M_{(2,2)}.$$

The (outer) coproduct on Qsym satisfies

$$\Delta(M_\alpha) = \sum_{\alpha=\beta\gamma} M_\beta \otimes M_\gamma,$$

where β or γ may be empty, and $\beta\gamma$ denotes concatenation. The counit on Qsym maps every element to its constant term.

We have described Qsym as a connected graded bialgebra. Any such bialgebra has an antipode and is thus a Hopf algebra (see, e.g., [7, Lemma 2.1]). See [7, 11] for a description of the antipode for Qsym.

1.2. **Peak sets and odd compositions.** The *peak set* of a permutation $\pi \in S_n$ is defined to be $\{i \in [n-1] \mid \pi(i-1) < \pi(i) > \pi(i+1)\}$. We will describe a bijection between the collection of all peak subsets of $[n-1]$ and the set $\mathrm{Odd}(n)$ of *odd compositions* of n, that is, compositions with only odd parts. Given $\alpha = (2i_1+1, \ldots, 2i_k+1) \in \mathrm{Odd}(n)$, define

$$\widehat{\alpha} = (\overbrace{2,\ldots,2}^{i_1}, 1, \overbrace{2,\ldots,2}^{i_2}, 1, \ldots, \overbrace{2,\ldots,2}^{i_k}, 1).$$

If $\widehat{\alpha} = (\overbrace{1,\ldots,1}^{j_1}, 2, \overbrace{1,\ldots,1}^{j_2}, 2, \ldots, \overbrace{1,\ldots,1}^{j_l}, 2, 1, \ldots, 1)$, then let

$$\widehat{S}_\alpha = \left\{ \sum_{m=1}^{s} (j_m + 2) \mid 1 \le s \le l \right\}.$$

For example, if $\alpha = (3,1,5)$ then $\widehat{\alpha} = (2,1,1,2,2,1)$ and $\widehat{S}_\alpha = \{2,6,8\}$. Note that $\widehat{S}_\alpha = \emptyset$ when $\alpha = (1,1,\cdots,1)$. The map $\alpha \mapsto \widehat{S}_\alpha$ is clearly a bijection between $\mathrm{Odd}(n)$ and the peak subsets of $[n-1]$.

The following identities are easy to verify:

$$S_\alpha = [n-1] \backslash (\widehat{S}_\alpha \cup (\widehat{S}_\alpha - 1)), \tag{1.5}$$

$$l(\alpha) = (\# \text{ of 1's in } \widehat{\alpha}), \tag{1.6}$$

$$\frac{n - l(\alpha)}{2} = |\widehat{S}_\alpha| = (\# \text{ of 2's in } \widehat{\alpha}) = l(\widehat{\alpha}) - l(\alpha). \tag{1.7}$$

1.3. **The peak algebra.** The *peak algebra* Π (over $\mathbb{Q}$) is the subspace of Qsym spanned by the *peak functions*

$$\theta_\alpha = \sum_{\substack{\beta \in \mathrm{Comp}(n): \\ \widehat{S}_\alpha \subset S_\beta \cup (S_\beta + 1)}} 2^{l(\beta)} M_\beta, \tag{1.8}$$

for $\alpha \in \mathrm{Odd}(n)$, $n \geq 0$. In Stembridge's work [19], θ_α is the weight enumerator for all enriched P-partitions of a labeled chain of size $|\alpha|$ with peak set $\widehat{S}_\alpha$. The set $\{\theta_\alpha \mid \alpha \in \mathrm{Odd}(n)\}$ is a linear basis for $\Pi_n = \Pi \cap \mathsf{Qsym}_n$, so $\dim_{\mathbb{Q}} \Pi_n$ equals the nth Fibonacci number z_n, where $z_1 = z_2 = 1$. As its name suggests, the peak algebra is a subalgebra of Qsym [19, Theorem 3.1]. In fact, Π is closed under coproduct and is therefore a Hopf subalgebra of Qsym [3, Theorem 2.2].

For $S \subset \mathbb{N}$, define

$$\Lambda(S) = \{i \in S \mid i \neq 1, i - 1 \notin S\}.$$

Equivalently, if S is the descent set of a permutation then $\Lambda(S)$ is the peak set of that permutation. For $\alpha \in \mathrm{Comp}(n)$, define $\Lambda(\alpha)$ to be the unique composition in $\mathrm{Odd}(n)$ satisfying $\widehat{S}_{\Lambda(\alpha)} = \Lambda(S_\alpha)$.

Consider the linear map $\Theta : \mathsf{Qsym} \to \Pi$ given by

$$\Theta(F_\alpha) = \theta_{\Lambda(\alpha)} \tag{1.9}$$

for all $\alpha \in \mathrm{Comp}(n)$, $n \geq 0$. Stembridge [19] introduced Θ as a natural way to relate the ordinary and enriched theories of P-partitions, and he showed that this map is an algebra homomorphism. Bergeron et al. [3] later observed that Θ preserves coproducts and is therefore a Hopf algebra homomorphism. In this paper, Θ is an important vehicle for transferring information about Qsym down to Π.

Example 1.2. Let $\alpha = (1, 3, 1, 2, 2, 2) \in \mathrm{Comp}(11)$ and $S = S_\alpha = \{1, 4, 5, 7, 9\}$. Then $\Lambda(S) = \{4, 7, 9\}$. To find $\Lambda(\alpha)$, use $\widehat{S}_{\Lambda(\alpha)} = \{4, 7, 9\}$ to obtain $\widehat{\Lambda(\alpha)} = (1, 1, 2, 1, 2, 2, 2, 1, 1)$. Thus $\Lambda(\alpha) = (1, 1, 3, 5, 1) \in \mathrm{Odd}(11)$.

2. A monomial basis for the peak algebra

Define the *monomial peak functions* η_α for $\alpha \in \mathrm{Odd}(n)$, $n \geq 0$, by the equivalent sets of relations

$$\eta_\alpha = \sum_{\beta \in \mathrm{Odd}(n): \widehat{S}_\beta \subset \widehat{S}_\alpha} (-1)^{|\widehat{S}_\alpha| - |\widehat{S}_\beta|} \theta_\beta, \tag{2.1}$$

$$\theta_\alpha = \sum_{\beta \in \mathrm{Odd}(n): \widehat{S}_\beta \subset \widehat{S}_\alpha} \eta_\beta. \tag{2.2}$$

For simplicity we let $\eta_n = \eta_{(n)}$, $M_n = M_{(n)}$, and so forth.

Proposition 2.1. *For any* $\alpha \in \mathrm{Odd}(n)$, $n \geq 0$, *we have*

$$\eta_\alpha \;=\; (-1)^{\frac{n-l(\alpha)}{2}} \sum_{\beta \in \mathrm{Comp}(n):\beta \leq \alpha} 2^{l(\beta)} M_\beta.$$

In particular, if $n = 2k+1$, *then* $\eta_n = (-1)^k 2M_n = (-1)^k 2p_n$, *where* p_n *is the power sum symmetric function of degree* n.

Proof. Using (2.1) and (1.8) we get

$$\eta_\alpha = (-1)^{|\widehat{S}_\alpha|} \sum_{\beta \in \mathrm{Comp}(n)} 2^{l(\beta)} M_\beta \left(\sum_{\substack{\gamma \in \mathrm{Odd}(n): \\ \widehat{S}_\gamma \subset (\widehat{S}_\alpha \cap (S_\beta \cup (S_\beta + 1))}} (-1)^{|\widehat{S}_\gamma|} \right).$$

Now $\widehat{S}_\alpha \cap (S_\beta \cup (S_\beta+1)) = \emptyset \Leftrightarrow S_\beta \cup (S_\beta + 1) \subset [n-1]\backslash \widehat{S}_\alpha \Leftrightarrow S_\beta \subset [n-1]\backslash(\widehat{S}_\alpha \cup (\widehat{S}_\alpha - 1)) = S_\alpha$. Thus the inner sum is 1 if $\beta \leq \alpha$ and 0 otherwise. □

Proposition 2.2. *For any* $\alpha \in \mathrm{Odd}(n)$, $n \geq 0$, *we have*

$$\eta_\alpha = 2\,(-1)^{\frac{n-l(\alpha)}{2}} \sum_{\beta \in \mathrm{Comp}(n)} (-1)^{|S_\beta \backslash S_\alpha|} F_\beta.$$

Proof. Substitute (1.3) into Proposition 2.1; the computation is routine. In the final step, use the identity $|S_\beta| + |S_\alpha \cap S_\beta| = |S_\beta \backslash S_\alpha| + |S_\beta \cap S_\alpha| + |S_\beta \cap S_\alpha|$. □

We now define a monomial counterpart to the operator Λ and give a useful expression of $\Theta(M_\alpha)$ in terms of η_α. If α is any composition whose last part is odd, then there is a unique factorization $\alpha = \alpha_{(1)} \cdot \alpha_{(2)} \cdots \alpha_{(l)}$ such that the last part of each $\alpha_{(j)}$ is odd and all other parts are even. In this case, let

$$\vartheta(\alpha) = (|\alpha_{(1)}|, |\alpha_{(2)}|, \ldots, |\alpha_{(l)}|).$$

We see no obvious way to define $\vartheta(\alpha)$ if the last part of α is even, so we leave $\vartheta(\alpha)$ undefined in this case. It follows from the definition of ϑ that

$$\vartheta^{-1}(\alpha) = \{\beta \in \mathrm{Comp}(n) \mid \alpha \leq \beta \leq \widehat{\alpha}\} \tag{2.3}$$

for all $\alpha \in \mathrm{Odd}(n)$.

Example 2.3. Let $\alpha = (2,3,1,5,4,2,3)$. From the factorization $\alpha = (2,3) \cdot (1) \cdot (5) \cdot (4,2,3)$ we see that $\vartheta(\alpha) = (5,1,5,9)$.

Theorem 2.4. *Let $\alpha \in \mathrm{Comp}(n)$. Then*

$$\Theta(M_\alpha) = \begin{cases} (-1)^{l(\widehat{\vartheta(\alpha)})-l(\alpha)}\eta_{\vartheta(\alpha)} & \text{if the last part of } \alpha \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\alpha \in \mathrm{Odd}(n)$ then $\Theta(M_\alpha) = (-1)^{l(\widehat{\alpha})-l(\alpha)}\eta_\alpha$ and $\Theta(M_{\widehat{\alpha}}) = \eta_\alpha$.

Proof. For each *peak* subset $R \subset [n-1]$, let

$$\mathcal{A}_R = \{T \subset [n-1] \mid T \supset S_\alpha \text{ and } \Lambda(T) = R\}.$$

Define $\theta_{\widehat{S}_\gamma} = \theta_\gamma$ for all $\gamma \in \mathrm{Odd}(n)$. Using (1.3) and (1.9) we obtain

$$\Theta(M_\alpha) = (-1)^{|S_\alpha|} \sum_{\substack{R \subset [n-1]: \\ R \text{ a peak set}}} \left(\sum_{T \in \mathcal{A}_R} (-1)^{|T|} \right) \theta_R.$$

Let c_R denote the coefficient of θ_γ in the expression above. By convention $c_R = 0$ if $\mathcal{A}_R = \emptyset$.

Suppose that $S_\alpha = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_m, b_m]$, where each $[a_i, b_i]$ is a maximal interval in S_α and $b_0 := 0 < b_1 < \cdots < b_m < a_{m+1} := n$. Let

$$U = \bigcup_{i=0}^{k} ([b_i, a_{i+1}] \cap \{b_i + 2j\}_{j=1}^{\infty}).$$

It is straightforward to verify that $U = \widehat{S}_{\vartheta(\alpha)}$ if the last part of α is odd.

Claim 1. Let $R \subset [n-1]$ be a peak set such that $R \not\subset U$. Then $c_R = 0$.

Proof of Claim. It follows from the definition of Λ that if $R \not\subset \bigcup_{i=0}^m [b_i + 2, a_{i+1}]$ then $\mathcal{A}_R = \emptyset$, in which case $c_R = 0$. So suppose that $R \subset \bigcup_{i=0}^m [b_i + 2, a_{i+1}]$ and $R \not\subset U$. Then there exists some i such that $(R \cap [b_i + 2, a_{i+1}]) \backslash \{b_i + 2j\}_{j=1}^{\infty} \neq \emptyset$. Let t be the smallest element in this set, and let $r = \max(\{b_i\} \cup \{r' \in R \mid r' < t\})$.

The set $\mathcal{A}_R$ consists of all sets of the form $A \cup B$, where $A \subset [n] \backslash [r, t-1]$ and $B = [r+1, s]$ for some $r \leq s < t-1$. For each A there are $t-r-1$ (which is even) possibilities for B, and $|B|$ is even in exactly half of the cases, so the sum $\sum (-1)^{|A \cup B|}$ over these B is 0. It then follows that $c_R = 0$. This completes the proof of the claim.

Claim 2. Suppose that the last part of α is even. Then $\Theta(M_\alpha) = 0$.

Proof of Claim. By Claim 1 it suffices to show that $c_R = 0$ for all $R \subset U$. Fix $R \subset U$. Let $r = \max(R \cup \{b_m\})$. Note that the last part of α is $n - b_m - 1$, and r and b_m have the same parity. Apply the second part of the argument from the proof of Claim 1 (with $t = n+2$) to show that $c_R = 0$. The proof of Claim 2 is complete.

Assume from now on that the last part of α is odd, or equivalently, $n - b_m$ is even. For any $R = \{r_1 < \cdots < r_l\} \subset U$, let

$$\mathsf{T}_R = [r_1, s_1] \cup [r_2, s_2] \cup \cdots \cup [r_l, s_l], \tag{2.4}$$

where $r_{l+1} = n+2$ and $s_i = \max(\{r_i\} \cup \{b_j \mid b_j \in [r_i + 1, r_{i+1} - 1]\})$ for $1 \le i \le l$. One verifies that T_R is the unique minimal element of $\mathcal{A}_R$ under inclusion.

Claim 3. $\Theta(M_\alpha) = (-1)^{|S_\alpha|} \sum_{R \subset U} (-1)^{|\mathsf{T}_R|} \theta_R$.

Proof of Claim. Let $s_0 = 0$. The set $\mathcal{A}_R$ consists of all sets of the form

$$\mathsf{T}_R \cup [s_0 + 1, t_0] \cup [s_1 + 1, t_1] \cup \cdots \cup [s_l + 1, t_l],$$

where $t_i \in [s_i, r_{i+1} - 2]$. For each $0 \le j \le l$, let D_j be the set of all $T \in \mathcal{A}_R$ of the form

$$T = \mathsf{T}_R \cup [s_j + 1, t_j] \cup [s_{j+1} + 1, t_{j+1}] \cup \cdots \cup [s_l + 1, t_l],$$

where $t_j \in [s_j + 1, r_{j+1} - 2]$. For every choice of $t_{j+1}, t_{j+2}, \ldots, t_l$ there are $r_{j+1} - s_j - 2$ (which is even, since $R \subset U$) choices for t_j, with half of these choices giving rise to T's of even size and the other half giving T's of odd size. Thus we get $\sum_{T \in D_j} (-1)^{|T|} = 0$, and so

$$c_R = (-1)^{|\mathsf{T}_R|} + \sum_{j=0}^{l} \sum_{T \in D_j} (-1)^{|T|} = (-1)^{|\mathsf{T}_R|}.$$

Claim 4. $(-1)^{|\mathsf{T}_R|} = (-1)^{|\mathsf{T}_U| + |U \setminus R|} = (-1)^{|S_\alpha| + |U \setminus S_\alpha| + |U| - |R|}$.

Proof of Claim. The second equality is obvious since $R \subset U$ and $\mathsf{T}_U = S_\alpha \cup U$. Let A be the elements in $U \setminus R$ that are in $[a_i - 1, a_i]$ for some i, and let B be those that are in $[b_i + 2, a_{i+1} - 2]$ for some i. Then we have $U \setminus R = A \,\dot\cup\, B$ (disjoint union), and using (2.4) we can check that $\mathsf{T}_U = \mathsf{T}_{R \cup A} \,\dot\cup\, B$, which yields

$$(-1)^{|\mathsf{T}_U| + |U \setminus R|} = (-1)^{|\mathsf{T}_{R \cup A}| + |A|}.$$

Suppose that $a \in A \cap [a_i - 1, a_i]$ for some i. Let $r = \max(\{b_i\} \cup (R \cap [b_i, a-1]))$. Comparing $\mathsf{T}_{R \cup A}$ and $\mathsf{T}_{R \cup (A \setminus a)}$, we find that the two sets agree on $[n] \setminus [r+1, a-1]$, but that $\mathsf{T}_{R \cup A} \cap [r+1, a-1] = \emptyset$ and $\mathsf{T}_{R \cup (A \setminus a)} \cap [r+1, a-1] = [r+1, a-1]$. The size of $[r+1, a-1]$ is odd since r and a have the same parity, and hence

$$(-1)^{|\mathsf{T}_{R \cup A}| + |A|} = (-1)^{|\mathsf{T}_{R \cup (A \setminus a)}| + |A \setminus a|}.$$

To finish the proof of the claim, replace A by $A\backslash a$ and repeat this argument until we are left with $A = \emptyset$.

The previous two claims, together with $U = \widehat{S}_{\vartheta(\alpha)}$, give

$$\Theta(M_\alpha) = (-1)^{|U\backslash S_\alpha|} \sum_{R \subset U} (-1)^{|U|-|R|} \theta_R = (-1)^{|U\backslash S_\alpha|} \eta_{\vartheta(\alpha)}.$$

The proof of the theorem will be complete once we show that $|U\backslash S_\alpha| = l(\widehat{\vartheta(\alpha)}) - l(\alpha)$. By definition of U, we have $U\backslash S_\alpha = \{a_i \mid a_i = b_{i-1} + 2j \text{ some } j \geq 1\}$. Thus each $a_i \in S_\alpha$ contributes an even part $a_i - b_{i-1}$ to α. Furthermore, every even part arises as $a_i - b_{i-1}$ for some i. Now we have $|U\backslash S_\alpha| = |U| - (\#\text{ even parts in } \alpha) = |\widehat{S}_{\vartheta(\alpha)}| - l(\alpha) + (\#\text{ odd parts in } \alpha) = (\#\text{ 2's in } \widehat{\vartheta(\alpha)}) - l(\alpha) + (\#\text{ 1's in } \widehat{\vartheta(\alpha)}) = l(\widehat{\vartheta(\alpha)}) - l(\alpha)$. □

Corollary 2.5. *Let α and β be two odd compositions. Then*

$$\eta_\alpha \eta_\beta = \sum_{(\tau,\gamma)} \eta_{\vartheta(\gamma)},$$

where the sum is over all quasi-shuffles of α and β whose last part is odd.

Proof. As in Definition 1.1, we interpret α and β as maps $\alpha : X \to \mathbb{P}$ and $\beta : Y \to \mathbb{P}$ for two disjoint chains X and Y. By (1.4), Theorem 2.4, and the fact that Θ is an algebra map, we have

$$\eta_\alpha \eta_\beta = \sum_{(\tau,\gamma)} (-1)^{l(\alpha)+l(\beta)-l(\gamma)+l(\widehat{\alpha})+l(\widehat{\beta})-l(\widehat{\vartheta(\gamma)})} \eta_{\vartheta(\gamma)},$$

where the sum is over all quasi-shuffles of α and β ending in an odd part. For each (τ, γ), we have $l(\alpha) + l(\beta) - l(\gamma) = \#\{(x, y) \in X \times Y \mid \tau(x) = \tau(y)\} = l(\widehat{\alpha}) + l(\widehat{\beta}) - l(\widehat{\vartheta(\gamma)})$, so the coefficient of $\eta_{\vartheta(\gamma)}$ is 1. □

Example 2.6.

$$\begin{aligned} \eta_{(5)}\eta_{(1,3)} &= \eta_{\vartheta(5,1,3)} + \eta_{\vartheta(6,3)} + \eta_{\vartheta(1,5,3)} + \eta_{\vartheta(1,3,5)} \\ &= \eta_{(5,1,3)} + \eta_{(9)} + \eta_{(1,5,3)} + \eta_{(1,3,5)}. \end{aligned}$$

The quasi-shuffle $(1, 8)$ is missing from the indices in the first row because its last part is even.

Corollary 2.7. *Let α be an odd composition. Then*

$$\Delta(\eta_\alpha) = \sum_{\alpha=\beta\gamma} \eta_\beta \otimes \eta_\gamma.$$

Proof. By Theorem 2.4 and the fact that Θ is a coalgebra map, we have

$$\Delta(\eta_\alpha) = \sum_{\alpha=\beta\gamma} (-1)^{l(\widehat{\alpha})-l(\alpha)+l(\beta)-l(\widehat{\beta})+l(\gamma)-l(\widehat{\gamma})}\eta_\beta \otimes \eta_\gamma.$$

If $\alpha = \beta\gamma$, then $l(\alpha) + l(\beta) = l(\gamma)$ and $l(\widehat{\alpha}) + l(\widehat{\beta}) = l(\widehat{\gamma})$, and so the coefficient of $\eta_\beta \otimes \eta_\gamma$ in the sum is 1. □

Example 2.8.

$$\Delta(\eta_{(3,1,5)}) = 1 \otimes \eta_{(3,1,5)} + \eta_{(3)} \otimes \eta_{(1,5)} + \eta_{(3,1)} \otimes \eta_{(5)} + \eta_{(3,1,5)} \otimes 1.$$

Proposition 2.9. *Let* $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathrm{Odd}(n)$ *and* $S : \mathsf{Qsym} \to \mathsf{Qsym}$ *be the antipode. Then*

$$S(\eta_{(\alpha_1,\alpha_2,\dots,\alpha_k)}) = (-1)^n \eta_{(\alpha_k,\dots,\alpha_2,\alpha_1)}. \tag{2.5}$$

Proof. An explicit formula for the antipode in terms of the peak functions is given in [4]. In that paper, peak functions are indexed by **cd**-words. We defer a discussion of how to translate between odd compositions and **cd**-words until Section 5. In terms of odd compositions, [4, Prop. 1.1] shows that

$$S(\theta_{(\alpha_1,\dots,\alpha_k)}) = (-1)^n \theta_{(\alpha_k,\dots,\alpha_1)}.$$

Using (2.1) and the fact that S is linear, we get the desired formula. □

3. Restriction of Θ to Π_n

Billera et al. [4] showed that the restriction map $\Theta|_{\Pi_n}$ is diagonalizable, with eigenvalues all given by powers of 2. Here we determine the effect of this map on the monomial peak functions. We then describe a new basis of eigenvectors for $\Theta|_{\Pi_n}$. In Section 4 this basis plays an important role in determining the algebraic structure of Π.

Proposition 3.1. *If* $\alpha \in \mathrm{Odd}(n)$, *then*

$$\Theta(\eta_\alpha) = \sum_{\beta \in \mathrm{Odd}(n) : \beta \leq \alpha} 2^{l(\beta)} \eta_\beta.$$

Proof. Using Proposition 2.2 and (2.2), we get

$$\Theta(\eta_\alpha) = 2(-1)^{|\widehat{S}_\alpha|} \sum_{\beta \in \mathrm{Odd}(n)} \eta_\beta \left(\sum_{\substack{T \subset [n-1]: \\ \Lambda(T) \supset \widehat{S}_\beta}} (-1)^{|T \setminus S_\alpha|} \right).$$

Let $\beta \in \mathrm{Odd}(n)$ be such that $\beta \not\leq \alpha$. Choose some $i \in S_\beta \setminus S_\alpha$. Then i is the position of a 1 in $\widehat{\beta}$, and hence $i, i+1 \notin \widehat{S}_\beta$. It follows that for any $T \subset [n-1]$, we have $\Lambda(T \setminus \{i\}) \supset \widehat{S}_\beta \Leftrightarrow \Lambda(T \cup \{i\}) \supset \widehat{S}_\beta$. The terms in the inner sum can therefore be paired off so that their sum is 0. We may now replace the range of the outer sum by $\{\beta \in \mathrm{Odd}(n) \mid \beta \leq \alpha\}$.

Next, fix $\beta = (\beta_1, \ldots, \beta_k) \in \mathrm{Odd}(n)$ and $T \subset [n-1]$ such that $\beta \leq \alpha$ and $\Lambda(T) \supset \widehat{S}_\beta$. We wish to show that $|T \backslash S_\alpha| = |\widehat{S}_\alpha|$. Since $\beta \leq \alpha$, there is a factorization $\alpha = \alpha_{(1)} \cdots \alpha_{(k)}$ such that $|\alpha_{(j)}| = \beta_j$ for all j. Let $S_\beta = \{s_1, s_2, \cdots, s_{k-1}\}_<$, $s_0 = 0$, and $s_k = n-1$. Define

$$T_j = \{t - s_{j-1} \mid t \in T \text{ and } s_{j-1} < t < s_j\}$$

for $1 \leq j \leq k$. Then $|T \backslash S_\alpha| = \sum_{j=1}^{k} |T_j \backslash S_{\alpha_{(j)}}|$. Together, the facts that $T \subset [n-1]$ and $\Lambda(T_j) = \widehat{S}_{\beta_j} = \{2, 4, 6, \ldots, \beta_j - 1\}$ imply $T_j = \widehat{S}_{\beta_j}$. Since $\alpha_{(j)}$ is odd, exactly half of the elements in $S_{\alpha(j)}$ are even, and so $|T_j \backslash S_{\alpha(j)}| = |T_j| - |S_{\alpha_{(j)}}|/2 = \frac{\beta_j - l(\alpha_{(j)})}{2} = |\widehat{S}_{\alpha_{(j)}}|$. It follows that $|T \backslash S_\alpha| = \sum_{j=1}^{k} |\widehat{S}_{\alpha_{(j)}}| = |\widehat{S}_\alpha|$.

We have shown that

$$\Theta(\eta_\alpha) = \sum_{\beta \in \mathrm{Odd}(n): \beta \leq \alpha} 2\, |\{T \subset [n-1] \mid \Lambda(T) \supset \widehat{S}_\beta\}|\, \eta_\beta.$$

For fixed β, let $\widehat{S}_\beta = \{b_1, b_2, \cdots, b_m\}_<$, $b_0 = 0$, and $b_{m+1} = n+1$. It is immediate from the definition of Λ that

$$\{T \subset [n-1] \mid \Lambda(T) \supset \widehat{S}_\beta\} = \{\widehat{S}_\beta \cup R \mid R \subset \prod_{i=0}^{m} [b_i + 1, b_{i+1} - 2]\}.$$

The cardinality of this set is $2^{n+1-2(m+1)} = 2^{n-2|\widehat{S}_\beta|-1} = 2^{l(\beta)-1}$, as desired. □

Remark 3.1. Proposition 3.1 suggests the following calculation involving the inverse map of $\Theta|_{\Pi_n}$. Let μ_n, $n \geq 0$, denote the Möbius function for $(\mathrm{Odd}(n), \leq)$. Applying Möbius inversion to Proposition 3.1 gives

$$(\Theta|_{\Pi_n})^{-1}(\eta_\alpha) = 2^{-l(\alpha)} \sum_{\beta \in \mathrm{Odd}(n): \beta \leq \alpha} \mu_n(\beta, \alpha) \eta_\beta, \tag{3.1}$$

for any $\alpha \in \mathrm{Odd}(n)$, $n \geq 0$. The values $\mu_n(\beta, \alpha)$ are signed products of Catalan numbers; see [16, Chapter 3 Exercise 52] for additional facts concerning the poset $(\mathrm{Odd}(n), \leq)$.

Define $\kappa(\beta, \alpha)$ for $\beta \leq \alpha \in \mathrm{Odd}(n)$, $n \geq 0$, by the following recursion:

$$\kappa(\beta, \alpha) = \begin{cases} 1 & \text{if } \beta = \alpha, \\ \dfrac{1}{2^{l(\alpha)-l(\beta)} - 1} \displaystyle\sum_{\gamma \in \mathrm{Odd}(n): \beta < \gamma \leq \alpha} \kappa(\gamma, \alpha) & \text{if } \beta < \alpha. \end{cases}$$

In addition, define ε_α and $\rho(\beta,\alpha)$ by the equivalent sets of relations

$$\varepsilon_\alpha = \sum_{\beta\in\mathrm{Odd}(n):\beta\le\alpha} \kappa(\beta,\alpha)\eta_\beta, \tag{3.2}$$

$$\eta_\alpha = \sum_{\beta\in\mathrm{Odd}(n):\beta\le\alpha} \rho(\beta,\alpha)\varepsilon_\beta. \tag{3.3}$$

Note that ρ is the inverse of κ in the *incidence algebra* of $(\mathrm{Odd}(n),\le)$ for each n.

A complete analysis of the spectrum of $\Theta|_{\Pi_n}$ was given in [4, Theorem 3.1]. Here we are able to determine the spectrum using a different basis of eigenvectors.

Theorem 3.2. *For any $n\ge 0$ and $\alpha\in\mathrm{Odd}(n)$, we have $\Theta(\varepsilon_\alpha) = 2^{l(\alpha)}\varepsilon_\alpha$. Consequently, the set $\{\varepsilon_\alpha \mid \alpha\in\mathrm{Odd}(n)\}$ is a basis of eigenvectors of the map $\Theta|_{\Pi_n}$, and the eigenvalues are 2^{n-2k} with multiplicity $\binom{n-k-1}{k}$, $0\le k\le\lfloor\frac{n}{2}\rfloor$.*

Proof. The defining recursion for κ gives the identity

$$\sum_{\gamma\in\mathrm{Odd}(n):\beta\le\gamma\le\alpha} \kappa(\beta,\gamma) = 2^{l(\alpha)-l(\beta)}\kappa(\beta,\alpha).$$

By Proposition 3.1, equation (3.2), and the previous identity, we get

$$\Theta(\varepsilon_\alpha) = \sum_{\beta\in\mathrm{Odd}(n):\beta\le\alpha}\left(2^{l(\beta)}\sum_{\gamma\in\mathrm{Odd}(n):\beta\le\gamma\le\alpha}\kappa(\beta,\gamma)\right)\eta_\beta = 2^{l(\alpha)}\varepsilon_\alpha.$$

It is clear from (3.2) that $\{\varepsilon_\alpha \mid \alpha\in\mathrm{Odd}(n)\}$ is a basis for Π_n. From (1.7) it follows that $2^{l(\alpha)} = 2^{n-2k} \Leftrightarrow k = (\#\text{ of 2's in }\hat{\alpha})$. Therefore the multiplicity of 2^{n-2k} is the number of compositions of $n-1$ with k parts equal to 2 and all other parts equal to 1. □

4. Structure theorems

For the rest of this paper, the dual of a graded vector space $V = \bigoplus_{n\ge0} V_n$ will always mean the *graded dual* $V^* = \bigoplus_{n\ge0} V_n^*$, where V_n^* denotes the ordinary dual of V_n. Each V_n will always be finite dimensional here, so $V^*\cong V$. If $v_1,\dots,v_s$ is a basis for some V_n then let $v_1^*,\dots,v_s^*$ denote its dual basis, so that under the pairing between V^* and V we have $\langle v_i^*, v_j\rangle = \delta_{i,j}$. If in addition V is a graded Hopf algebra, then V^* is also a graded Hopf algebra, with product and coproduct satisfying

$$\langle v^*u^*, w\rangle = \langle v^*\otimes u^*, \Delta(w)\rangle \tag{4.1}$$

$$\langle \Delta(v^*), u\otimes w\rangle = \langle v^*, uw\rangle. \tag{4.2}$$

We remind the reader that when the composition (n) appears as an index, we omit the parentheses; for example η_n means $\eta_{(n)}$. It follows immediately from (4.1) and Corollary 2.7 that

$$\eta^*_\alpha \eta^*_\beta = \eta^*_{\alpha\beta}, \tag{4.3}$$

for all odd compositions α and β. Consequently, $\{\eta^*_{2k+1} \mid k \geq 0\}$ is a free generating set for the associative algebra Π^*.

Theorem 4.1. *The Hopf algebra Π^* is the graded free associative algebra on generators $\{\varepsilon^*_{2k+1} \mid k \geq 0\}$, with $\deg(\varepsilon^*_n) = n$ and coproduct satisfying $\Delta(\varepsilon^*_n) = \varepsilon^*_n \otimes 1 + 1 \otimes \varepsilon^*_n$.*

Proof. Since Θ is an algebra map, any product of the form $\varepsilon_\alpha \varepsilon_\beta$ is again an eigenvector with eigenvalue $2^{l(\alpha)+l(\beta)}$. Thus the expansion of $\varepsilon_\alpha \varepsilon_\beta$ in the ε-basis only involves terms ε_γ for which $l(\gamma) = l(\alpha) + l(\beta)$. In particular, the coefficient of ε_n is zero for all α, β and n odd. Equivalently, $\langle \Delta(\varepsilon^*_n), \varepsilon_\alpha \otimes \varepsilon_\beta \rangle = \langle \varepsilon^*_n, \varepsilon_\alpha \varepsilon_\beta \rangle = 0$ unless $\alpha = \emptyset$ and $\beta = (n)$, or $\alpha = (n)$ and $\beta = \emptyset$. Thus,

$$\Delta(\varepsilon^*_n) = \varepsilon^*_n \otimes 1 + 1 \otimes \varepsilon^*_n,$$

that is, ε^*_n is a primitive element of Π^*.

The dual form of (3.3) yields

$$\varepsilon^*_n = \sum_{\alpha \in \mathrm{Odd}(n) : \alpha \geq (n)} \rho((n), \alpha) \eta^*_\alpha = \eta^*_n + \sum_{\alpha \in \mathrm{Odd}(n) : \alpha > (n)} \rho((n), \alpha) \eta^*_\alpha. \tag{4.4}$$

Hence by induction

$$\eta^*_n \in \mathbb{Q} \left\langle \eta^*_1, \eta^*_3, \ldots, \eta^*_{n-2}, \varepsilon^*_n \right\rangle = \mathbb{Q} \left\langle \varepsilon^*_1, \varepsilon^*_3, \ldots, \varepsilon^*_n \right\rangle$$

for all n odd, so Π^* is generated by $\{\varepsilon^*_{2k+1} \,|\, k \geq 0\}$. This generating set is free since $\{\eta^*_{2k+1} \,|\, k \geq 0\}$ is free and $\deg(\varepsilon^*_{2k+1}) = 2k+1 = \deg(\eta^*_{2k+1})$ for all $k \geq 0$. □

Given an odd composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, let

$$\tau^*_\alpha = \varepsilon^*_{\alpha_1} \cdots \varepsilon^*_{\alpha_k}.$$

By Theorem 4.1, $\{\tau^*_\alpha \mid \alpha \in \mathrm{Odd}(n)\}$ is a basis for Π^*_n. Elements of the dual basis in Π_n are denoted by τ_α.

A *Lyndon word* on a totally ordered alphabet is one which is lexicographically strictly smaller than all of its nontrivial cyclic permutations. Let L be the set of Lyndon words on the totally ordered alphabet $1 < 3 < 5 < \ldots$. We think of L as a subset of all odd compositions.

Corollary 4.2. *The $\mathbb{Q}$-algebra Π is a free commutative algebra on the generating set $\{\tau_\alpha \,|\, \alpha \in L\}$. Among this set are the symmetric functions*

$(-1)^k 2p_{2k+1}$ *for all* $k \geq 0$, *and hence* Π *is a free module over Schur's* Q*-function algebra.*

Proof. An equivalent way of stating Theorem 4.1 is that Π^* is the *concatenation Hopf algebra* over $\mathbb{Q}$ in the noncommuting variables $\varepsilon_1^*, \varepsilon_3^*, \varepsilon_5^*, \ldots$ The coproduct in a concatenation Hopf algebra is adjoint to the *shuffle product* [14, Section 1.5]; in other words Π is a *shuffle algebra.* It is well-known that a shuffle algebra is freely generated by its subset of Lyndon words [14, Theorem 6.1(i)], which in our case is the set $\{\tau_\alpha \mid \alpha \in L\}$. This proves the first claim.

From $\eta_\alpha^* \eta_\beta^* = \eta_{\alpha\beta}^*$ and (4.4) we get

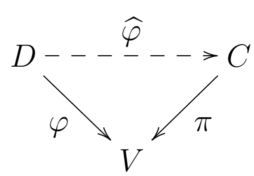
$$\tau_\beta^* = \sum_{\alpha \in \mathrm{Odd}(n) : \beta \leq \alpha} \rho'(\beta, \alpha) \eta_\alpha^*$$

for any $\beta \in \mathrm{Odd}(n)$; the exact form of $\rho'(\beta, \alpha)$ is not important here except in the obvious case $\rho'((n),(n)) = 1$ for all n odd. By duality,

$$\eta_\alpha = \sum_{\beta \in \mathrm{Odd}(n) : \beta \leq \alpha} \rho'(\beta, \alpha) \tau_\beta,$$

which reduces to the identity $\eta_n = \tau_n$ when $\alpha = (n)$. If n is odd then $(n) \in L$, and so $\{\eta_{2k+1} \mid k \geq 0\} \subset \{\tau_\alpha \mid \alpha \in L\}$. By Proposition 2.1, $\eta_{2k+1} = (-1)^k 2p_{2k+1}$. The proof is complete upon noting that the symmetric functions p_{2k+1}, $k \geq 0$, freely generate Schur's Q-function algebra [10]. □

Next we describe a coalgebra grading which makes Π into a cofree graded coalgebra. Our discussion follows [1, Section 6]. Let V be a vector space over $\mathbb{Q}$. A graded coalgebra $C = \bigoplus_{n \geq 0} C^n$, together with a linear map $\pi : C \to V$ satisfying $\pi(C^n) = 0$ for $n \neq 1$, is said to be a *cofree graded coalgebra on* V if for any graded coalgebra $D = \bigoplus_{n \geq 0} D^n$ and linear map $\varphi : D \to V$ such that $\varphi(D^n) = 0$ for $n \neq 1$ there is a unique graded coalgebra map $\widehat{\varphi}$ making the following diagram commute:



Of course C is unique up to isomorphism of graded coalgebras. The standard construction of a cofree graded coalgebra on V endows the vector space

$$Q(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

with the *deconcatenation coproduct*

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n)$$

and counit $\epsilon(v_1 \otimes \cdots \otimes v_n) = 0$ for $n \geq 1$.

For $k \geq 0$ let Π^k be the vector space spanned by $\{\eta_\alpha \mid l(\alpha) = k\}$. It is clear from Corollary 2.7 that

$$\Delta(\Pi^k) \subseteq \bigoplus_{i=0}^{k} \Pi^k \otimes \Pi^{k-i}.$$

In other words, $\Pi = \bigoplus_{k\geq 0} \Pi^k$ is a coalgebra grading.

Theorem 4.3. *The grading* $\Pi = \bigoplus_{k\geq 0} \Pi^k$ *and the coproduct* Δ *give* Π *the structure of a cofree graded coalgebra on* Π^1.

Proof. Let $V = \Pi^1$ and $\varphi : \Pi \to V$ be the standard projection onto this graded component. By the universal property, the induced map $\widehat{\varphi} : \Pi \to Q(V)$ must satisfy $\widehat{\varphi}(f) = \varphi^{\otimes k}\Delta^{(k-1)}(f)$ for all $f \in \Pi^k$. When $f = \eta_\alpha$, say with $\alpha = (\alpha_1, \ldots, \alpha_k)$, then by Corollary 2.7

$$\widehat{\varphi}(\eta_\alpha) = \eta_{\alpha_1} \otimes \cdots \otimes \eta_{\alpha_k}.$$

It is clear that the map $\psi : V^{\otimes k} \to \Pi^k$ given by

$$\psi(\eta_{\beta_1} \otimes \cdots \otimes \eta_{\beta_k}) = \eta_{(\beta_1,\ldots,\beta_k)}$$

is the two-sided inverse of $\widehat{\varphi}$, and so Π and $Q(V)$ are isomorphic as graded coalgebras. □

5. Flag-enumeration in Eulerian posets

A poset P is said to be *graded* if it has a unique minimal and maximal element, denoted by $\hat{0}$ and $\hat{1}$ respectively, and every maximal chain (totally ordered set) from $\hat{0}$ to $\hat{1}$ has the same length. For any $x \in P$, the length of any maximal chain from $\hat{0}$ to x is called the *rank* of x, denoted $r(x)$. The rank of P is defined to be $r(\hat{1})$. If $x \leq y \in P$, the *interval* from x to y in P is the subposet $[x, y] = \{z \in P \mid x \leq z \leq y\}$.

A graded poset P is *Eulerian* if its Möbius function satisfies $\mu(x, y) = (-1)^{r(y)-r(x)}$ whenever $x \leq y$. Equivalently, P is Eulerian if every interval in P has the same number of elements of odd rank as it does even rank. The face lattices of convex polytopes form an important family of Eulerian posets.

Let P be a graded poset of rank n and let $\alpha \in \mathrm{Comp}(n)$. The *flag number* $f_\alpha(P)$ is the number of chains in P of size $l(\alpha) - 1$ whose elements have rank set S_α. The map $\alpha \mapsto f_\alpha(P)$ is called the *flag f-vector* of P. We can view f_α as a linear functional, or *chain operator*, on the vector space spanned by isomorphism classes of all graded posets (setting $f_\alpha(P) = 0$ if $r(\hat{1}) \neq |\alpha|$).

Let $A_0 = \mathbb{Q}$, and for $n \geq 1$ let A_n be the vector space consisting of all homogeneous linear forms $\sum_{\alpha \in \mathrm{Comp}(n)} c_\alpha f_\alpha$ with coefficients in $\mathbb{Q}$. The vector space $A = \bigoplus_{n \geq 0} A_n$ has the structure of a free associative algebra, where multiplication is given by the rule $f_\alpha f_\beta = f_{\alpha\beta}$. See [5] for details.

We may identify A with the Hopf algebra Qsym^*, also called the Hopf algebra of *non-commutative symmetric functions* [8], via the association $f_\alpha \leftrightarrow M^*_\alpha$. Under this identification, there is a canonical isomorphism between Π^* and the quotient Hopf algebra $A/I_{\mathcal{E}}$, where $I_{\mathcal{E}}$ is the (Hopf) ideal consisting of the linear forms which vanish on all Eulerian posets [2, Theorem 5.4]. The quotient algebra $A/I_{\mathcal{E}}$ was first studied by Billera and Liu [5]. Elements of $A/I_{\mathcal{E}}$ are thought of as chain operators on Eulerian posets.

Various aspects of the dualities described above can be expressed in terms of the generating function

$$F(P) := \sum_{\alpha \in \mathrm{Comp}(n)} f_\alpha(P) M_\alpha \tag{5.1}$$

for P a graded poset of rank n. The definition of $F(P)$ is due to Ehrenborg [7]. An immediate consequence of [2, Theorem 5.4] is that $F(P) \in \Pi$ whenever P is Eulerian, in which case representing $F(P)$ in terms of some fixed basis of Π determines the corresponding dual basis in $A/I_{\mathcal{E}}$. For example, if P is Eulerian of rank n then

$$F(P) = \sum_{\alpha \in \mathrm{Odd}(n)} \theta^*_\alpha(P)\theta_\alpha = \sum_{\alpha \in \mathrm{Odd}(n)} \eta^*_\alpha(P)\eta_\alpha$$

where $\theta^*_\alpha(P) := \theta^*_\alpha(F(P))$ and $\eta^*_\alpha(P) := \eta^*_\alpha(F(P))$.

The numbers $\theta^*_\alpha(P)$ are closely related to a common invariant on Eulerian posets known as the **cd**-index, defined as follows. Let **c** and **d** be noncommuting variables of degrees 1 and 2, respectively. Given a **cd**-monomial w of degree $n-1$, consider the composition $\beta \in \mathrm{Comp}(n)$ obtained by changing each "**c**" in w to a "1" and each "**d**" to a "2", and then appending a single "1" at the end. Define α_w to be the unique composition in $\mathrm{Odd}(n)$ satisfying $\widehat{\alpha_w} = \beta$.

Example 5.1. Let $w = \mathbf{dcddc}$. Then $\widehat{\alpha_w} = 212211$, so $\alpha_w = 351$.

We define the **cd**-index of an Eulerian poset P of rank n to be the polynomial

$$\sum_w 2^{\frac{n-l(\alpha_w)}{2}+1}\theta^*_{\alpha_w}(P)w,$$

where the sum is over all **cd**-monomials of degree $n-1$. This polynomial appears quite different from the standard definition of **cd**-index (see [15]), though Billera et al. [4] have shown that these definitions are in fact equivalent. Thus, knowing the **cd**-index of P is equivalent to knowing the representation of $F(P)$ in the basis θ_α.

There is a surprising connection between the **cd**-index and a conjecture of Charney and Davis [6]. Their conjecture is a discrete analog of a conjecture of Hopf on the sign of the Euler characteristic of an even dimensional, closed Riemannian manifold with nonpositive curvature. Babson observed that in the special case of order complexes (i.e. simplicial complexes whose faces are the chains of $P\backslash\{\hat{0},\hat{1}\}$ for some poset P), the Charney-Davis conjecture asserts the following:

Conjecture 5.2 (Charney-Davis Conjecture for Posets)**.** Let P be a poset of rank $2k+1$ which is Eulerian and Cohen-Macaulay (such a poset is called Gorenstein*). Then the coefficient of $\mathbf{d}^k$ in the **cd**-index of P is nonnegative. Equivalently, $\theta^*_{2k+1}(P) \geq 0$.

Stanley [15] has made the following stronger conjecture:

Conjecture 5.3 (Gorenstein* Conjecture)**.** Let P be a Gorenstein* poset of rank n. Then all of the coefficients of the **cd**-index of P are nonnegative. Equivalently, $\theta^*_\alpha(P) \geq 0$ for all $\alpha \in \mathrm{Odd}(n)$.

Stanley proved Conjecture 5.3 in the case when P is the face lattice of a convex polytope. A step towards a general proof might be to study the quantities $\eta^*_\alpha(P)$. It is easy to verify that these numbers are the coefficients (up to a factor of a power of 2) of Reading's *Charney-Davis index* [13], introduced in connection with the Gorenstein* and Charney-Davis conjectures. In the present context, it is natural to view the quantities $\eta^*_\alpha(P)$ as Eulerian analogues of the flag numbers $f_\alpha(P) = M^*_\alpha(P)$, considering the analogy between the two bases η_α and M_α.

Conjecture 5.4. Let P be a Gorenstein* poset of of rank n. Then $\eta^*_\alpha(P) \geq 0$ for all $\alpha \in \mathrm{Odd}(n)$.

The dual form of (2.2) expresses each η^*_α as a sum of θ^*_β's. In particular, we have the identities $\eta^*_{2k+1} = \theta^*_{2k+1}$ for all $k \geq 0$. Now it is easy to see that

$$\text{Conjecture 5.3} \Rightarrow \text{Conjecture 5.4} \Rightarrow \text{Conjecture 5.2.}$$

We will show that the last two conjectures are actually equivalent.

Lemma 5.5. *Let α and β be odd compositions and let P be an Eulerian poset. Then*
$$\eta^*_{\alpha\beta}(P) = \sum_{x \in P} \eta^*_\alpha([\hat{0}, x])\eta^*_\beta([x, \hat{1}]).$$

Proof. Using the fact that Ehrenborg's map F is a coalgebra homomorphism from the incidence Hopf algebra of graded posets onto Qsym [7], we obtain
$$\begin{aligned} \eta^*_{\alpha\beta}(P) &= \eta^*_{\alpha\beta}(F(P)) = \langle \eta^*_\alpha \eta^*_\beta, F(P)\rangle = \langle \eta^*_\alpha \otimes \eta^*_\beta, \Delta(F(P))\rangle \\ &= \sum_{x \in P} \eta^*_\alpha([\hat{0}, x])\eta^*_\beta([x, \hat{1}]). \end{aligned}$$
□

Remark 5.1. A calculation similar to that in the proof of Lemma 5.5 leads to (more complicated) convolution formulas for $\theta^*_{\alpha\beta}(P)$. See [13] and [20] for statements and alternate proofs of these formulas.

Theorem 5.6. *Let $\mathcal{F}$ be a family of Eulerian posets that is closed under taking intervals, and suppose that $\eta^*_{2k+1}(P) \geq 0$ for all $k \geq 0$ and $P \in \mathcal{F}$. Then $\eta^*_\alpha(P) \geq 0$ for all odd compositions α and $P \in \mathcal{F}$.*

This theorem is essentially due to Reading [13, Theorem 1].

Proof. Let P be any Eulerian poset and $\alpha = (\alpha_1, \ldots, \alpha_k)$ be any odd composition. Repeated applications of Lemma 5.5 yields

$$\eta^*_\alpha(P) = \sum_{\hat{0}=x_0<x_1<\cdots<x_k=\hat{1}} \prod_{i=1}^{k} \eta^*_{\alpha_i}([x_{i-1}, x_i]), \tag{5.2}$$

from which the result follows. □

Corollary 5.7. *Conjectures 5.2 and 5.4 are equivalent.*

Proof. Dualizing (2.2) yields the identities $\eta^*_{2k+1} = \theta^*_{2k+1}$ for all $k \geq 0$. Now apply Theorem 5.6, taking $\mathcal{F}$ to be the family of all Gorenstein* posets. □

Stanley's work suggests that Gorenstein* posets might be the largest "natural" class of posets on which the functionals θ^*_α are nonnegative; see [15, Theorem 2.1]. It would be of interest to determine the largest class of Eulerian posets on which the functionals η^*_α are nonnegative.

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