A signed analog of the Birkhoff transform

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Abstract

We construct a family of posets, called signed Birkhoff posets, that may be viewed as signed analogs of distributive lattices. Our posets are generally not lattices, but they are shown to posses many combinatorial properties corresponding to well known properties of distributive lattices. They have the additional virtue of being face posets of regular cell decompositions of spheres. We relate the zeta polynomial of a signed Birkhoff poset to Stembridge's enriched order polynomial and give a combinatorial description the **cd**-index of a signed Birkhoff poset in terms of peak sets of linear extensions of an associated labeled poset. Our description is closely related to a result of Billera, Ehrenborg, and Readdy's expressing the **cd**-index of an oriented matroid in terms of the flag f-vector of the underlying geometric lattice.

Key words: Distributive lattice, Eulerian poset, flag f-vector, **cd**-index, enriched P-partition, quasisymmetric function.
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1 Introduction

This paper introduces a signed analog of the classical construction of a distributive lattice $J(P)$ from a finite poset P. Beginning with the work of Birkhoff [10], distributive lattices have been well-studied from a combinatorial viewpoint. Nowadays they are often analyzed in conjunction with notions such as P-partitions, linear extensions, and EL-labelings; see, e.g., [34, Chapter 3]. Our construction will give rise to a family of Eulerian posets that are amenable to similar types of analyses. Stembridge's enriched P-partitions [36] turn out to play a role in the enumeration theory of these posets that is analogous to

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the role of Stanley's P-partitions [31] for distributive lattices. Our enumerative analysis is motivated by the work of Billera, Ehrenborg, and Readdy on the **cd**-index of oriented matroids [6]. While the posets that we construct are not directly related to face lattices of oriented matroids, the flag vectors of these two classes of posets are seen to have many similar features.

Given a positive integer n and a poset P on the set $\{1, 2, \ldots, n\}$ partially ordered by $\leq_P$, let $\pm P$ be the poset on $\{\pm 1, \ldots, \pm n\}$ partially ordered so that $p <_{\pm P} q$ if and only if $|p| <_P |q|$. We define the *signed Birkhoff transform of* P to be the poset $B(P)$ consisting of the filters (upper order ideals) X of $\pm P$ such that if p is a minimal element in X then $-p \notin X$; order these filters by inclusion. Let $\widehat{B}(P)$ denote the poset $B(P)$ with a unique maximal element added. Any poset of the form $B(P)$ or $\widehat{B}(P)$ is called a *signed Birkhoff poset.* [2] These posets are the main objects of study in this paper. We will repeat these definitions in Section 2 after reviewing some poset terminology. Examples will then follow.

We summarize the main results.

In Section 3 we describe a "pairing procedure" that allows one to recover P uniquely (up to isomorphism) from $B(P)$. This is analogous in part to Birkhoff's fundamental theorem for finite distributive lattices, which asserts that every finite distributive lattice L is isomorphic to the poset of order ideals of the subposet of join irreducibles of L. Presently lacking in this analogy is an intrinsic characterization of signed Birkhoff posets that avoids reference to an underlying poset P. Interestingly, $\widehat{B}(P)$ is not a lattice unless P is an antichain (Proposition 2.3), so the pairing procedure does not involve lattice notions such as join irreducibility.

Section 4 deals with shellability properties of signed Birkhoff posets. We show that the edge-labeling of $\widehat{B}(P)$ induced by a natural labeling of P is an *EL*-labeling and a dual R-labeling (Theorem 4.1). This implies that $\widehat{B}(P)$ is Gorenstein* (i.e., Eulerian and Cohen-Macaulay) for every P. The Gorenstein* property is also a consequence of the fact that $B(P)$ is the face poset of a regular shellable cell decomposition of a sphere (Theorem 4.7). This result, first established by Billera and the author, is proved here by showing that $\widehat{B}(P)$ admits a recursive coatom ordering (Theorem 4.5), then invoking a theorem of Björner's on cellular interpretations of posets [12].

Section 5 deals with enumerative aspects of signed Birkhoff posets. Let P_0 denote the poset P with a unique minimal element added. We establish the

[2] To our knowledge, there is no direct connection between signed Birkhoff posets and the hyperoctahedral analogs of posets, called signed posets, introduced by Reiner [28].

identity (Theorem 5.1)

$$2F_{\widehat{B}(P)^*} = \widetilde{K}_{P_0} \tag{1.1}$$

relating Ehrenborg's F-quasisymmetric function (which encodes the flag f-vector) of the dual poset $\widehat{B}(P)^*$ to the weight enumerator for enriched P_0-partitions. This fundamental identity follows easily from Stembridge's original work on enriched P-partitions as well as from Bergeron, Mykytiuk, Sottile, and van Willigenburg's theory of Eulerian Pieri operators [4, Section 7]. The latter work is relevant because of the close connection between the signed Birkhoff transform and the doubled réseau of a distributive lattice. A corollary of (1.1) is that the zeta polynomial of $\widehat{B}(P)$ is one-half the enriched order polynomial of P_0. This fact is relevant to unimodality questions for certain polynomials arising in the theory of enriched P-partitions.

We also derive from (1.1) a combinatorial interpretation of the **cd**-index of $\widehat{B}(P)$ in terms of peak sets of linear extensions of P_0 (Theorem 5.6). Our description implies that the **cd**-index of $\widehat{B}(P)$ is coefficient-wise maximized when P is an antichain and minimized when P is a chain. There is an elegant reformulation of (1.1) that directly relates the **cd**-index of a signed Birkhoff poset to the flag f-vector of its underlying distributive lattice (Theorem 5.15). This reformulation is essentially identical to the expression provided by Billera, Ehrenborg, and Readdy [6] relating the **cd**-index of an oriented matroid to the flag f-vector of its geometric lattice of flats (Theorem 5.14).

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2 Definitions, examples, and preliminary results

2.1 Poset terminology.

We briefly review key definitions related to posets. See [34, Chapter 3] for further background. All posets in this paper are assumed to be finite unless otherwise indicated.

Let P be a poset on n elements partially ordered by $\leq_P$. A *filter* or *upper order ideal* of P is a subset $X \subseteq P$ such that if $p \in X$ and $p \leq_P q$ then $q \in X$. A filter is uniquely determined by its set of minimal elements, which we call its *generators*. Denote by $\langle p_1, \ldots, p_m \rangle$ the filter with generators $p_1, \ldots, p_m$. A filter with only one generator is called *principal*. The *dual poset* of P is the poset P^* consisting of the elements of P with partial order satisfying $x \leq_{P^*} y$ if and only if $y \leq_P x$. A *linear extension* of P is a linear ordering $p_1, \ldots, p_n$ of the elements of P such that $p_i <_P p_j$ implies $i < j$. The set of sequences in $P^{\times n}$ corresponding to linear extensions of P is denoted by $\mathcal{L}(P)$. Say that P is a *lattice* if any two elements in P have a least upper bound and a greatest lower bound. Call P a *chain* if any two of its elements are comparable under $\leq_P$. Call P an *antichain* if no two distinct elements are comparable under $\leq_P$. If $p \leq_P q$, the *(closed) interval* $[p, q]$ is the poset on the set $\{r : p \leq_P r \leq_P q\}$ with partial order induced by $\leq_P$.

For a positive integer m, let $[m] := \{1, 2, \ldots, m\}$. Let $[0] := \emptyset$. Say that P is *graded* of rank m if it has a unique maximal element $\hat{1}$, a unique minimal element $\hat{0}$, and every maximal chain in P has length m. In this case there is a unique rank function $rk : P \to \{0, 1, \ldots, m\}$ satisfying $rk(\hat{0}) = 0$, $rk(\hat{1}) = m$, and $rk(y) = rk(x)+1$ whenever y covers x, written $x \lessdot y$. If P has rank m and $S \subseteq [m-1]$, let $f_S(P)$ be the number of chains $p_1 <_P p_2 <_P \cdots <_P p_k$ of P such that $\{rk(p_1), \ldots, rk(p_k)\} = S$. The vector $(f_S(P) : S \subseteq [m-1])$ is called the *flag f-vector* of P. A graded poset P is called *Eulerian* if its Möbius function satisfies $\mu_P(p, q) = (-1)^{rk(p,q)}$ for every $p \leq_P q$, where $rk(p, q) := rk(q) - rk(p)$. It is called *Cohen-Macaulay* (over $\mathbb{Q}$) if the homology of the order complex (i.e., simplicial complex of chains) of every open interval in P vanishes below the top dimension. A poset that is Eulerian and Cohen-Macaulay is called *Gorenstein**. The *face poset* $P(\Gamma)$ of a finite regular cell complex Γ is the poset of cells of Γ, along with the empty cell, ordered by inclusion of their closures. Let $\widehat{P}(\Gamma)$ denote the poset $P(\Gamma)$ with a unique maximal element added. When no confusion will arise we may refer to $\widehat{P}(\Gamma)$ as the face poset of Γ.

Assumptions. Assume throughout this paper that $n > 0$ is fixed and P is a *naturally labeled* poset on $[n]$ partially ordered by $\leq_P$; that is, $p <_P q$ implies $p < q$ as integers. Let P_0 denote the naturally labeled poset obtained from P by adding a unique minimal element labeled 0.

2.2 The signed Birkhoff transform.

The *Birkhoff transform of* P is the poset (distributive lattice) $J(P)$ consisting of the filters of P ordered by reverse inclusion.[3] Let $\pm P$ be the poset on $\{\pm 1, \ldots, \pm n\}$ ordered so that $p <_{\pm P} q$ if and only if $|p| <_P |q|$. A *signed P-filter* is a filter X of $\pm P$ such that if p is a generator of X then $-p \notin X$.

[3] Usually $J(P)$ is defined as the poset of (lower) order ideals of P under inclusion, rather than as the filters under reverse inclusion; these two definitions yield isomorphic posets. Filters will be more convenient for our purposes.

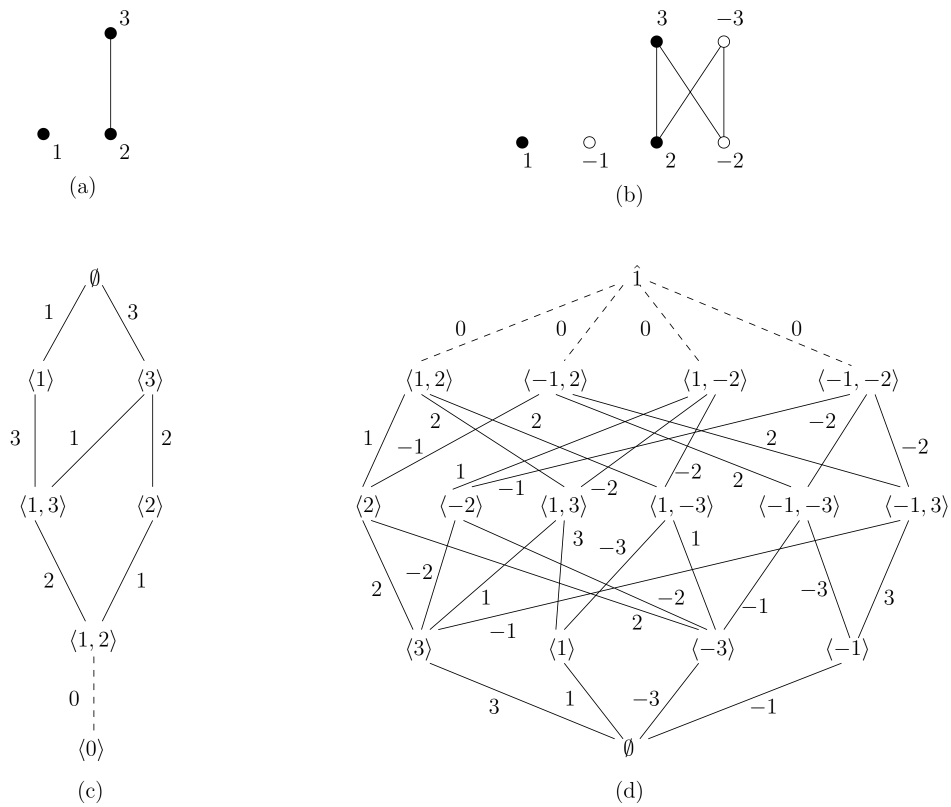

Fig. 1. (a) A naturally labeled poset P; (b) the labeled poset $\pm P$; (c) the Birkhoff transforms $J(P)$ (without $\langle 0\rangle$) and $J(P_0)$ (with $\langle 0\rangle$), with edge-labeling induced by P; (d) the signed Birkhoff transform $B(P)$ (without $\hat{1}$) and $\widehat{B}(P)$ (with $\hat{1}$), with edge-labeling induced by P.

The *signed Birkhoff transform* of P is the poset $B(P)$ consisting of the set of signed P-filters ordered by inclusion.

One could define the signed Birkhoff transform more abstractly without first identifying P with $[n]$. This identification is made here for notational convenience and without loss of generality. It is evident that the isomorphism type of $B(P)$ depends only on the isomorphism type of P.

Let $\widehat{B}(P)$ denote the poset $B(P)$ with a unique maximal element $\hat{1}$ added. Any poset of the form $B(P)$ or $\widehat{B}(P)$ is called a *signed Birkhoff poset.* For clarity we sometimes call $\widehat{B}(P)$ a *graded signed Birkhoff poset* (cf. Proposition 2.4).

Figure 1 illustrates both the ordinary and signed Birkhoff transforms of a three element poset. Let us also point out two interesting families of examples:

Example 2.1. If P is an n-element chain, then $B(P)$ is isomorphic to the face poset of a regular cell decomposition of the $(n-1)$-sphere with exactly two cells in each dimension. Such a poset is sometimes called a ladder.

Example 2.2. If P is an n-element antichain, then $B(P)$ is isomorphic to the face poset of the boundary of an n-dimensional cross-polytope.

Some familiar properties of the Birkhoff transform carry over to the signed transform without much difficulty. For instance, as with the identity $J(P \sqcup Q) \cong J(P) \times J(Q)$, it is straightforward to show that

$$B(P \sqcup Q) \cong B(P) \times B(Q), \tag{2.1}$$

where $\sqcup$ and $\times$ denote, respectively, the disjoint union and cartesian product for posets.

Unlike the class of distributive lattices, the class of signed Birkhoff posets is not closed under taking intervals. For instance, the poset in Figure 1(d) has several intervals that are isomorphic to the Boolean lattice of rank 3, which itself is not a signed Birkhoff poset. The following result points to another significant difference between these two classes of posets.

Proposition 2.3. *$\widehat{B}(P)$ is a lattice if and only if P is an antichain.*

Proof. The "if" statement is clear from Example 2.2 and the fact that the face poset of the cross-polytope is a lattice. To prove the converse, suppose that for some $p \in P$ the set $\{q_1, \ldots, q_k\}$ of elements covering p is not empty. The following are cover relations in $B(P)$: $\langle q_1, \ldots, q_k \rangle \lessdot \langle p \rangle$, $\langle q_1, \ldots, q_k \rangle \lessdot \langle -p \rangle$, $\langle -q_1, q_2, \ldots, q_k \rangle \lessdot \langle p \rangle$, $\langle -q_1, q_2, \ldots, q_k \rangle \lessdot \langle -p \rangle$. Thus $\langle p \rangle$ and $\langle -p \rangle$ do not have a greatest lower bound. $\square$

In the sequel it will be useful to relate ordinary and signed Birkhoff transforms via the order-reversing surjection $\varphi : \widehat{B}(P) \to J(P_0)$ defined by

$$\varphi(X) = \begin{cases} \{|p| : p \in X\} & \text{if } X \in B(P), \\ P_0 & \text{if } X = \hat{1}. \end{cases}$$

Note that φ restricts to a map from $B(P)$ onto $J(P)$.

The cover relations in $J(P)$ are precisely those relations of the form $A \cup \{p\} < A$ for some maximal element p of $P \backslash A$. Thus $J(P)$ is graded of rank n with rank function given by $rk(A) = n - \#A$. The analogous assertions for signed Birkhoff posets are easily verified:

Proposition 2.4. *The cover relations in $B(P)$ are precisely those relations of the form $X < X \cup \langle p \rangle$ such that p and $-p$ are maximal elements of $\pm P \backslash X$ or, equivalently, $|p|$ is a maximal element of $P \backslash \varphi(X)$. Thus $\widehat{B}(P)$ is a graded poset of rank $n+1$ with rank function given by $rk(X) = \#\varphi(X)$.*

It is a basic property of the Birkhoff transform that a sequence $(p_1, \ldots, p_n) \in P^{\times n}$ is in $\mathcal{L}(P)$ if and only if $\{p_1, \ldots, p_n\} < \{p_2, \ldots, p_n\} < \cdots < \{p_n\} < \emptyset$ is a maximal chain of $J(P)$. By Proposition 2.4, if $c = \{\emptyset = X_0 \lessdot X_1 \lessdot \cdots \lessdot X_n\}$ is a maximal chain of $B(P)$ then there exists a sequence $\lambda(c) = (p_1, \ldots, p_n) \in (\pm P)^{\times n}$ such that $X_i = X_{i-1} \cup \langle p_i \rangle$ for all i. Such sequences can be characterized as signed linear extensions of P:

Proposition 2.5. *Let $\pi \in (\pm P)^{\times n}$. Then $\pi = \lambda(c)$ for some (necessarily unique) maximal chain c of $B(P)$ if and only if $\pi = \varepsilon\sigma$ for some $(\varepsilon, \sigma) \in \{\pm 1\}^{\times n} \times \mathcal{L}(P)$.*

Proof. The proof is immediate from Proposition 2.4 and the ensuing discussion. □

Remark 2.6. The *doubled réseau* $\delta J(P)$ studied by Bergeron, et al. in [4] is the directed graph obtained by replacing each labeled edge $A \cup \{p\} \xrightarrow{p} A$ in the Hasse diagram of $J(P)$ with the two labeled edges $A \cup \{p\} \underset{-p}{\overset{p}{\rightrightarrows}} A$. In light of Proposition 2.5, we may view signed Birkhoff posets as "poset realizations" of doubled réseaux of distributive lattices. It is then possible to infer a direct connection between flag enumeration in $\widehat{B}(P)$ and weight enumeration of enriched P-partitions via the theory of Eulerian Pieri operators developed in [4, Section 7]; see Theorem 5.1 and Remark 5.2.

Remark 2.7. Simion [30] introduced the notion of an *f-q-order analog of a poset*. Let f be the element in the *incidence algebra* of $J(P_0)^*$ defined on any pair of filters $A \subseteq A'$ as the number of generators for A that are not generators for A'. One can easily check that the poset $\widehat{B}(P)$ together with the element f and the map $\varphi : \widehat{B}(P) \to J(P_0)^*$ constitute an f-q-order analog of $J(P_0)^*$ for $q = 2$, in the sense of [30]. (That paper also mentions another family of examples of f-q-order analogs of distributive lattices, for a different f and any $q \geq 1$, due to Stanley and Björner.) Some general results about f-q-order analogs are proved in that paper, but most of these results are not applicable in the present situation because f does not satisfy the condition referred to as "compatibility with respect to a shelling".

3 The pairing procedure.

An element of a lattice is called *join irreducible* if it covers exactly one element. Birkhoff's fundamental theorem on finite distributive lattices [34, Theorem 3.4.1 and Proposition 3.4.2] asserts that if P is the subposet of join irreducible elements of a finite distributive lattice L, then $L \cong J(P)$. Thus P is uniquely determined by L up to isomorphism. We describe an analogous procedure for recovering P from $B(P)$. Let $B = B(P)$. Define an equivalence relation on B by putting $X \equiv X'$ if and only if X and X' cover exactly the same set of elements, so in particular X and X' are of the same rank. Let $T_1, \ldots, T_m$ be the non-singleton equivalence classes in $B/\equiv$, indexed so that $i < j$ whenever the elements of T_i have rank greater than those of T_j. Our goal is to inductively construct posets $B_1, \ldots, B_m$ whose isomorphism types depend only on the isomorphism type of B and then show that $B_m \cong P^*$.

Lemma 3.1. *For every $p \in P$, we have $\langle p \rangle \equiv \langle -p \rangle$. Moreover, if $X \equiv X'$ and $X \neq X'$ then X and X' are principal filters.*

Proof. The first assertion is clear. Suppose that X and X' are distinct signed P-filters of the same rank and X is not principal. Then there is some generator p of X such that $p \notin X'$, and hence some filter containing p that is covered by X but not by X'. Thus, $X \not\equiv X'$. □

By the lemma, every T_i is the union of sets of the form $\{\langle p \rangle, \langle -p \rangle\}$. Fix a partition of T_1 into blocks of size two and let B_1 be the antichain consisting of these blocks.

Assume by induction that the poset B_{i-1} has been constructed for some $i > 1$. Given $X, X' \in T_i$, write $X \equiv_i X'$ provided that for every $j < i$ and $Y \in T_j$ we have $X < Y$ if and only if $X' < Y$. Each equivalence class in $T_i/\equiv_i$ has even size because $\langle p \rangle \equiv_i \langle -p \rangle$ for any p. Now partition each equivalence class in $T_i/\equiv_i$ arbitrarily into blocks of size two. Define the poset B_i by adjoining these two-element blocks to B_{i-1} and, for any such block $\{X, X'\}$ and any $\{Y, Y'\} \in B_{i-1}$, putting $\{X, X'\} <_{B_i} \{Y, Y'\}$ if and only if X and X' are both less than Y and Y'.

Example 3.2. Let $B = B(P)$ be the poset from Figure 1(d). Then the pairing procedure yields the following:

1. $T_1 = \{\langle 2 \rangle, \langle -2 \rangle\}$;
2. $T_2 = \{\langle 1 \rangle, \langle -1 \rangle, \langle 3 \rangle, \langle -3 \rangle\}$;
3. B_1 is the one-element antichain $\{\{\langle 2 \rangle, \langle -2 \rangle\}\}$;
4. $T_2/\equiv_2 \; = \{\{\langle 1 \rangle, \langle -1 \rangle\}, \{\langle 3 \rangle, \langle -3 \rangle\}\}$;
5. B_2 is the poset on the set $\{\langle 2 \rangle, \langle -2 \rangle\}, \{\langle 1 \rangle, \langle -1 \rangle\}, \{\langle 3 \rangle, \langle -3 \rangle\}$ with exactly one relation, $\{\langle 3 \rangle, \langle -3 \rangle\} <_{B_2} \{\langle 2 \rangle, \langle -2 \rangle\}$.

Note that B_2 is isomorphic to P^* via the map $\{\langle p \rangle, \langle -p \rangle\} \mapsto |p|$.

Theorem 3.3. *The pairing procedure, when applied to $B(P)$, always produces a poset that is isomorphic to P^*. Thus P is uniquely determined by $B(P)$ up to isomorphism.*

Proof. As before, let $B_1, \ldots, B_m$ be a sequence of posets obtained by applying the pairing procedure to $B(P)$. We claim that for each $i = 1, \ldots, m$, the isomorphism type of B_i does not depend on the choice of partition of the equivalence classes in $T_i/{\equiv_i}$ into two-element blocks. The claim is obvious for $i = 1$. Proceed by induction on i. For some $i > 1$, let $\{Y, Y'\} \in B_{i-1}$ and $X \in T_i$. If $X < Y$ then $X < Y'$ by definition of $\equiv$, so for every X' such that $X' \equiv_i X$ we get $X' < Y$ and $X' < Y'$ by definition of $\equiv_i$. This shows that given $X' \equiv_i X$ and $\{Y, Y'\} \in B_{i-1}$, either X and X' are both less than Y and Y', or X and X' are both incomparable with Y and Y'. The proves the inductive step.

It is now easy to show that $B_m \cong P^*$. According to the previous paragraph, we may assume $B_m = \{\{\langle p\rangle, \langle -p\rangle\} : p \in P\}$. The proof will be complete once we show that $\{\langle p\rangle, \langle -p\rangle\} <_{B_m} \{\langle q\rangle, \langle -q\rangle\}$ if and only if $|q| <_P |p|$. The forward implication is immediate from the definition of $<_{B_m}$. The reverse implication holds because the T_i's are indexed in order of decreasing rank. $\square$

4 Shellability and sphericity

4.1 EL-shellability.

An *edge-labeling* of a poset is a map from its cover relations to the integers. The edge-labeling of $J(P)$ *induced by* P is defined by mapping each cover relation $A \cup \{p\} \lessdot A$ to p. Similarly, the edge-labeling of $B(P)$ induced by P is defined by mapping the cover relation $X \lessdot X \cup \langle p\rangle$ to p. We extend this to an edge-labeling of $\widehat{B}(P)$ by mapping each cover relation of the form $X \lessdot \hat{1}$ to 0. Figure 1 illustrates induced edge-labelings.

Let λ be an edge-labeling of a graded poset Q. Given a chain $c = \{q_0 \lessdot q_1 \lessdot \cdots \lessdot q_m\}$ that is maximal in some interval $[q_0, q_m]$ in Q, say that c is *increasing* if its label-sequence $\lambda(c) := (\lambda(q_0, q_1), \ldots, \lambda(q_{m-1}, q_m))$ is a strictly increasing sequence, and say that c is *decreasing* if $\lambda(c)$ is a strictly decreasing sequence. Call λ an *R-labeling* if every interval I has a unique increasing chain, which we denote by a_I. Call λ an *EL-labeling* if it is an R-labeling and for every interval I, $\lambda(a_I)$ is lexicographically smaller than $\lambda(c)$ for any other maximal chain c of I. Call λ a *dual R-labeling* if it is an R-labeling of the dual poset Q^*. See [11] for further background.

If Q has an *EL*-labeling, then the lexicographic ordering of its maximal chains determines a *shelling* of the order complex of Q [11]. For this reason we call such a poset *EL-shellable*. The induced edge-labeling of $J(P)$ is well-known (and easily shown) to be an *EL*-labeling.

Theorem 4.1. *The induced edge-labeling of $\widehat{B}(P)$ is both an EL-labeling and a dual R-labeling.*

Corollary 4.2. *$\widehat{B}(P)$ is Gorenstein*.*

Proof of Corollary 4.2. The Cohen-Macaulay property follows from EL-shellability [11]. Furthermore, by a well known result of Stanley and Björner [11, Theorem 2.7] any R-labeling can be used to evaluate the Möbius function as follows: For every $X < Y$ in $\widehat{B}(P)$, the quantity $(-1)^{rk(X,Y)}\mu(X, Y)$ equals the number of decreasing chains in the interval $[X, Y]$; this number is 1 since we have a dual R-labeling, so $\widehat{B}(P)$ is Eulerian. □

We introduce some notation and establish a preliminary result before giving a proof of Theorem 4.1. Let λ be the induced edge-labeling of $B(P)$. If $S \subseteq \pm P$, we let $\pm S = S \cup \{-s : s \in S\}$ and $\mathsf{Max}(S)$ (respectively, $\mathsf{Min}(S)$) be the set of maximal (respectively, minimal) elements in S with respect to $<_{\pm P}$. Given $X < Y$ in $B(P)$, define $a_{[X,Y]}$ (respectively, $d_{[X,Y]}$) to be the chain $\{X = X_0 \lessdot X_1 \lessdot \cdots \lessdot X_m = Y\}$ characterized by the following property: for every $1 \leq j \leq m$ we have $X_j = X_{j-1} \cup \langle p_j \rangle$, where p_j is the smallest (respectively, largest) integer in $\mathsf{Max}(Y \backslash (\pm X_{j-1}))$. For instance, if $B(P)$ is the poset in Figure 1(d), and $I = [\emptyset, \langle 1, -2\rangle]$, then

$$a_I = \{\emptyset \lessdot \langle -3\rangle \lessdot \langle -2\rangle \lessdot \langle 1, -2\rangle\};$$

$$d_I = \{\emptyset \lessdot \langle 3\rangle \lessdot \langle 1, 3\rangle \lessdot \langle 1, -2\rangle\}.$$

Proposition 4.3. *In any interval I of $B(P)$, a_I is the unique increasing maximal chain and d_I is the unique decreasing maximal chain. Moreover, among all label-sequences of maximal chains in I, $\lambda(a_I)$ is lexicographically first and $\lambda(d_I)$ is lexicographically last.*

Proof. Let $(p_1, \ldots, p_m) = \lambda(a_I)$ and $[X, Y] = I$. It is clear from our choice of p_1 in the definition of a_I that $(p_1, \ldots, p_m)$ is the lexicographically smallest label-sequence, so we only need to show that a_I is the unique increasing chain in I. The following fact will be useful:

Claim: If $-p_i \in Y$ for some $i \in [m]$, then $p_i < 0$.

To see this, observe that since p_i is in $\mathsf{Max}(Y \backslash (\pm X_{i-1}))$, so is $-p_i$; hence, $p_i < -p_i$ (as integers) by definition of p_i, which proves the claim.

Let us first show that a_I is increasing. Fix $i \in [m-1]$. Since p_i is chosen before p_{i+1} when constructing a_I, it is not possible that $p_i <_{\pm P} p_{i+1}$. Suppose that $p_{i+1} <_{\pm P} p_i$. Then by definition of $<_{\pm P}$ we have $p_{i+1} <_{\pm P} -p_i$ and $|p_{i+1}| <_P |p_i|$. The former inequality implies $-p_i \in Y$; hence $p_i < 0$ by the claim. The latter inequality implies $|p_{i+1}| < |p_i|$ as P is naturally labeled, so $p_i < p_{i+1}$. Suppose next that p_i and p_{i+1} are incomparable in $\pm P$. Then it must be that p_i and p_{i+1} are both in $\mathsf{Max}(Y \backslash (\pm X_{i-1}))$. Since p_i was chosen before p_{i+1}, we have $p_i < p_{i+1}$. Thus a_I is increasing.

It remains to show that there is no other increasing chain in I. Suppose that $c = \{X = X'_0 \lessdot X'_1 \lessdot \cdots \lessdot X'_m = Y\}$ is a chain different from a_I. Let $S = \{p_1, \ldots, p_m\}$ and $S' = \{p'_1, \ldots, p'_m\}$, where $(p'_1, \ldots, p'_m) = \lambda(c)$.

If $S = S'$, then $\lambda(c)$ and $\lambda(a_I)$ are two different linear orderings of S; since $\lambda(a_I)$ is increasing, $\lambda(c)$ cannot be increasing as well. Suppose that $S \neq S'$. It is an easy consequence of Proposition 2.4 and the definition of λ that any two maximal chains in an interval in $B(P)$ have the same set of edge-labels, up to variation in signs. Therefore, $-p_i = p'_j$ for some $i, j \leq m$. This implies that p_i and $-p_i$ are both in Y, so Y has a generator q such that $q <_{\pm P} \pm p_i$. Since c is maximal, there exists an index k such that $j < k \leq m$ and $X'_k = X'_{k-1} \cup \langle q \rangle$; i.e., $q = p'_k$. The proof will be complete once we show that $p'_j > p'_k$ or, equivalently, $-p_i > q$. We have $-p_i > 0$ by the claim. Since P is naturally labeled, $-p_i = |-p_i| > |q| > q$. This completes the proof in the case of a_I.

In the case of d_I, it is clear that $\lambda(d_I) = (p_1, \ldots, p_m)$ is lexicographically last. The rest of the proof is analogous to the argument just given, with the technical claim modified to read: If $-p_i \in Y$ for some $i \in [m]$, then $p_i > 0$. We omit the remaining details to avoid needless repetition. □

Proof of Theorem 4.1 Let P' be the naturally labeled poset on $[\#P + 1]$ obtained from P by increasing each of its labels by 1 and then adjoining a unique minimal element labeled 1. The inclusion $P \to P'$, $p \mapsto p+1$, induces an isomorphism from $\widehat{B}(P)$ to the interval $I = [\emptyset, \langle 1 \rangle]$ of $B(P')$. This isomorphism clearly preserves the relative order of the induced edge labels. By Proposition 4.3, the edge-labeling of I is both an *EL*-labeling and dual *R*-labeling, so the same is true for the edge-labeling of $\widehat{B}(P)$.

Remark 4.4. The induced edge-labeling of $\widehat{B}(P)$ is generally *not* an *EL*-labeling for $\widehat{B}(P)^*$, as one can see from Figure 1(d).

4.2 Recursive coatom ordering. Sphericity

Let Q be a graded poset. A *coatom* of Q is an element covered by $\hat{1}$. Let *coat*(Q) denote the set of coatoms of Q. Following [16], we say that Q *admits a recursive coatom ordering* if its rank is 1, or if its rank is greater than 1 and there is an ordering $x_1, x_2, \ldots, x_m$ of its coatoms such that the following conditions hold:

(i) For all $j = 1, \ldots, m$, $[\hat{0}, x_j]$ admits a recursive coatom ordering in which the elements in $coat([\hat{0}, x_j]) \cap \Big(\cup_{i<j} coat([\hat{0}, x_i])\Big)$ come first.

(ii) For all $i < j$, if $y < x_i, x_j$ then there exist $k < j$ and $z \in Q$ such that $y \leq z \lessdot x_k, x_j$.

Theorem 4.5. *$\widehat{B}(P)$ admits a recursive coatom ordering.*

Proof. Since $\widehat{B}(P)$ is isomorphic to an interval in $B(P_0)$ (see proof of Theorem 4.1), it suffices to prove that every interval of $B(P)$ admits a recursive coatom ordering.

Let $g(X)$ denote the set of generators for any signed P-filter X. Given an interval $I = [\hat{0}_I, \hat{1}_I]$ in $B(P)$ and $p \in g(\hat{1}_I)$, define

$$I_p = \{X \in coat(I) : X \cup \langle p \rangle = \hat{1}_I\};$$

$$\epsilon_p = \{q \in \pm P : p \lessdot_{\pm P} q \text{ and } q \in g(\hat{0}_I)\};$$
$$\delta_p = \{|q| : q \in \pm P \text{ and } p \lessdot_{\pm P} q \text{ and } q \notin g(\hat{0}_I)\}.$$

Note that $I_p = \emptyset$ if and only if $p \in g(\hat{0}_I)$. Thus the sets I_p form a partition of $coat(I)$ into disjoint non-empty subsets as p ranges over $G_I := g(\hat{1}_I) \backslash g(\hat{0}_I)$. Note also that for each $p \in G_I$, we have $X \in I_p$ if and only if $g(X)$ is a disjoint union of the form

$$g(X) = (g(\hat{1}_I) \backslash \{p\}) \;\dot{\cup}\; \epsilon_p \;\dot{\cup}\; \sigma(X),$$

where $\sigma(X)$ is obtained from the (possibly empty) set δ_p by switching some element of δ_p to their negatives. Thus, for fixed $p \in G_I$, an element $X \in I_p$ is uniquely determined by $\sigma(X)$.

We now describe a way to order the coatoms of I that results in a recursive coatom ordering. For each $p \in G_I$, fix an arbitrary linear ordering of δ_p. Given $X \in I_p$, identify $\sigma(X)$ with the binary vector in $\{0,1\}^{\delta_p}$ given by $q \mapsto 0$ if $q \in \sigma(X)$ and $q \mapsto 1$ if $-q \in \sigma(X)$. Put the binary vectors $\{\sigma(X) : X \in I_p\}$ in ascending order according to the natural numbers they represent. This determines a linear ordering of the elements of I_p. Finally, keeping the internal ordering of each I_p, order the sets $I_p, p \in G_I$, arbitrarily. This procedure results in a linear ordering of the coatoms of I, which will be called a *generic ordering.* An example is provided after the proof.

We shall prove by induction on the rank of I that a generic ordering of $coat(I)$ is a recursive coatom ordering. There is nothing to prove when the rank is 1. Suppose that $I = [\hat{0}_I, \hat{1}_I]$ has rank greater than 1. Let $X_1, \ldots, X_m$ be a generic ordering of the coatoms of I. Write $[X_i] := [\hat{0}_I, X_i]$.

To verify condition (i) in the definition of recursive coatom ordering, it suffices to show that for any $i < j$, $coat([X_j]) \cap coat([X_i])$ is the union of sets of the form $[X_j]_q, q \in G_{[X_j]}$. Any such union can be arranged to be the beginning of a generic ordering for $[X_j]$, so by induction such a union is the beginning of a recursive coatom ordering for $[X_j]$.

Suppose that $X_i, X_j \in I_p$ for some $p \in G_I$. If X_i and X_j differ by exactly one generator, say $q \in \sigma(X_j)$ and $-q \in \sigma(X_i)$, then $coat([X_i]) \cap coat([X_j]) = [X_j]_q = [X_i]_{-q}$. If they differ by more than one generator, say $q, r \in \sigma(X_j)$ and $-q, -r \in \sigma(X_i)$ with $q \neq r$, then every coatom of $[X_j]$ will have either q or r as a generator, while every coatom of $[X_i]$ will have either $-q$ or $-r$ as a generator; hence $coat([X_j]) \cap coat([X_i]) = \emptyset$. Finally, if $X_i \in I_p$ and $X_j \in I_q$ for $p \neq q$, then it is easy to see that $coat([X_j]) \cap coat([X_i]) = [X_i]_q = [X_j]_p$.

Now we verify condition (ii). Suppose that $Y < X_i, X_j$ and $i < j$. First, suppose that $X_i, X_j \in I_p$ for some $p \in G_I$. Let s be the smallest element of δ_p on which $\sigma(X_i)$ and $\sigma(X_j)$ have different signs; here "smallest" is relative to the linear order that was chosen for δ_p in the generic ordering. Since $i < j$, the binary vector representing $\sigma(X_j)$ must have a 1 in position s, so $-s \in \sigma(X_j)$. Changing this 1 to a 0 yields a binary vector representing $\sigma(X_k)$ for some $k < j$; in particular, we have $[X_k]_s = [X_j]_{-s}$. Note that Y cannot have $-s$ as a generator because s is a generator for $X_i > Y$. Thus, there is some $Z \in [X_j]_{-s} = [X_k]_s$ such that $Y \leq Z$, as required. Finally, suppose that $X_i \in I_p$ and $X_j \in I_q$ for some $p \neq q$. Then it suffices to take $k = i$. Indeed, neither p nor q is a generator for Y, and so for every $Z \in [X_i]_q = [X_j]_p$ we have $Y \leq Z$. ☐

Example 4.6. Following the procedure given in the proof of the previous theorem, we exhibit a recursive coatom ordering of the interval $I = [\emptyset, \langle 1, -2 \rangle]$ in the poset from Figure 1(d). The coatoms of I are $X_1 = \langle -2 \rangle, X_2 = \langle 1, 3 \rangle$, and $X_3 = \langle 1, -3 \rangle$. We have $G_I = \{1, -2\}$, $I_1 = \{X_1\}$, $I_{-2} = \{X_2, X_3\}$, $\epsilon_1 = \epsilon_{-2} = \delta_1 = \emptyset$, $\delta_{-2} = \{3\}$, $\sigma(X_1) = \emptyset$, $\sigma(X_2) = \{3\}$, and $\sigma(X_3) = \{-3\}$. Inside I_1 we associate $\sigma(X_1)$ with the empty binary vector. Inside I_{-2} we associate $\sigma(X_2)$ with the binary vector 0 and $\sigma(X_3)$ with 1; hence the elements of I_{-2} must be ordered by X_2, X_3. Now order the two sets I_1 and I_{-2} arbitrarily, say I_{-2}, I_1. This results in the recursive coatom ordering X_2, X_3, X_1.

Note that certain valid recursive coatom orderings, such as X_2, X_1, X_3, are not obtainable by this procedure.

The recursive-coatom-ordering property is a purely combinatorial formulation of the concept of shellability for a regular cell complex. It also generalizes the notion of *EL*-shellability: For a graded poset Q,

$$Q \text{ is } EL\text{-shellable} \Longrightarrow Q^* \text{ admits a recursive coatom ordering.}$$

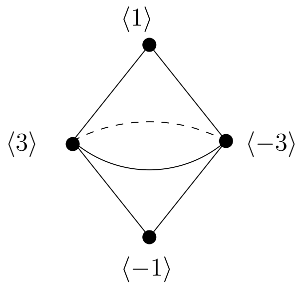

Fig. 2. A cell decomposition of the 2-sphere into four 0-cells, six 1-cells, and four 2-cells whose face poset is the signed Birkhoff poset in Figure 1(d).

These shelling properties make it possible to interpret intervals in signed Birkhoff posets (and their duals) as regular decompositions of spheres. Call a graded poset *thin* if every interval of rank 2 is Eulerian, that is, isomorphic to the Boolean lattice on 2 elements. Björner [12] showed that a graded poset Q of rank n is isomorphic to $\widehat{P}(\Gamma)$ for Γ a shellable regular cell decomposition of the $(n-2)$-sphere if and only if Q is thin and admits a recursive coatom ordering. Since the signed Birkhoff poset is Eulerian, it is a thin poset. Thus, Björner's theorem together with Theorem 4.1 and Theorem 4.5 yield

Theorem 4.7 (Billera and Hsiao). *Let $[X,Y]$ be an interval in $\widehat{B}(P)$ or $\widehat{B}(P)^*$. Then $[X,Y]$ is isomorphic to the face poset of a shellable regular cell decomposition of the $(rk(Y)-rk(X)-2)$-sphere.*

Figure 2 illustrates a cell complex whose face poset is the signed Birkhoff poset from Figure 1(d).

Remark 4.8. A proof that $B(P)$ is the face poset of a regular sphere was originally found by Billera and the author via an explicit geometric description of the cell decomposition. The geometric aspects of signed Birkhoff posets will be studied in greater detail elsewhere. We thank Sergey Fomin for pointing us to Björner's result.

5 Enumerative properties

5.1 Quasisymmetric generating functions.

Let $\mathcal{Q} = \bigoplus_{n\geq 0} \mathcal{Q}^n$ denote the *graded algebra of quasisymmetric functions* over $\mathbb{Q}$ in the variables $x_1, x_2, \ldots$ The vector space $\mathcal{Q}^n$ consists of those homogeneous power series in $\mathbb{Q}[[x_1, x_2, \ldots]]$ of degree n for which the coefficients of $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ and $x_{i_1}^{a_1} x_{i_1}^{a_2} \cdots x_{i_k}^{a_k}$ are equal whenever $i_1 < \cdots < i_k$ and $a_1, \ldots, a_k$ is a sequence of positive integers summing to n. Set $\mathcal{Q}^0 = \mathbb{Q}$. For each $n \geq 1$, the *fundamental basis* for $\mathcal{Q}^n$ is the linear basis consisting of the 2^{n-1} elements

$$L_S := \sum_{\substack{i_1 \le \cdots \le i_n: \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (S \subseteq [n-1]).$$

This notation suppresses the dependence of L_S on n. See [35, §7.19] for general background and references on quasisymmetric functions.

Let Q be a graded poset (with $\hat{0}$ and $\hat{1}$) of rank n with rank function rk. To study the flag enumerative invariants of Q, it will be useful to work with the following quasisymmetric generating function introduced by Ehrenborg [21]:

$$F_Q(x_1, x_2, \ldots) := \sum_{\substack{k \ge 1, \\ \hat{0} = t_0 \le t_1 \le \cdots \le t_{k-1} < t_k = \hat{1}}} x_1^{rk(t_0,t_1)} x_2^{rk(t_1,t_2)} \cdots x_k^{rk(t_{k-1},t_k)},$$

where the sum is over all multichains of Q from $\hat{0}$ to $\hat{1}$ in which $\hat{1}$ occurs exactly once. We simply write F_Q when there is no need to refer to the underlying variables. We review some essential facts about this generating function.

Setting the first m variables to 1 and the rest to 0 yields the zeta polynomial of Q:

$$F_Q(\underbrace{1, \ldots, 1}_{m}, 0, \ldots) = Z_Q(m). \tag{5.1}$$

For $m \ge 2$, $Z_Q(m)$ is the number of multichains $q_1 \le q_2 \le \cdots \le q_{m-1}$ in Q. Recall that the *descent set* of a sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of integers is defined by $Des(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$. If Q has an R-labeling λ, then

$$F_Q = \sum_c L_{Des(\lambda(c))}, \tag{5.2}$$

where the sum is over all maximal chains c of Q. In general, when Q does not necessarily have an R-labeling, the vector of coefficients of F_Q in the fundamental basis is the *flag h-vector* of Q. Let $\mathcal{A}(P)$ denote the set of *reverse P-partitions*, i.e., order-preserving maps from P to the positive integers. The *weight enumerator* for reverse P-partitions is the quasisymmetric function

$$K_P := \sum_{\sigma \in \mathcal{A}(P)} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Gessel [25] first studied quasisymmetric weight enumerators for more general objects called (P, ω)-partitions [31], the motivation being that these weight enumerators generalize Schur functions in a combinatorially useful way. It is easy to verify using (5.2) (see [35, page 359]) that

$$F_{J(P)} = K_P. \tag{5.3}$$

Theorem 5.1 on page 16 expresses a similar relationship between $F_{\widehat{B(P)}^*}$ and the weight enumerator for Stembridge's enriched P-partitions.

5.2 Enumeration in the peak algebra.

The *peak set* of a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ of integers is defined to be

$$Peak(\sigma) := \{i \in \{2, 3, \ldots, n-1\} : \sigma_{i-1} < \sigma_i > \sigma_{i+1}\}.$$

Let $Peak_n$ denote the set of all possible peak sets of sequences of length n. Thus, $S \in Peak_n$ if and only if (i) $1, n \notin S$ and (ii) $i \in S$ implies $i - 1 \notin S$. For each $S \in Peak_n$, the *peak function* $\theta_S \in \mathcal{Q}^n$ is defined by

$$\theta_S := 2^{\#S+1} \sum_{T \subseteq [n-1] : S \subseteq T \triangle (T+1)} L_T,$$

where $T \triangle U := (T \backslash U) \cup (U \backslash T)$ and $T + 1 := \{i + 1 : i \in T\}$. In the context of his theory of enriched P-partitions, Stembridge discovered that the peak functions are linearly independent and span a graded subalgebra $\Pi := \bigoplus_{n \geq 0} \Pi^n$ of $\mathcal{Q}$, called the *peak algebra* [36]. The peak functions had also appeared earlier in the work of Billey and Haiman [9].

Let $\pm\mathbb{P}$ be the linear order $-1 \prec +1 \prec -2 \prec +2 \prec -3 \prec +3 \prec \cdots$ on the set of non-zero integers. An *enriched P-partition* of a poset P is an order-preserving map $\sigma : P \to \pm\mathbb{P}$ such that if $\sigma(p) = \sigma(q)$ then $\sigma(p) > 0$. The *weight enumerator* for enriched P-partitions is the quasisymmetric function

$$\widetilde{K}_P := \sum_{\sigma} x_{|\sigma(1)|} x_{|\sigma(2)|} \cdots x_{|\sigma(n)|},$$

where the sum is over all enriched P-partitions. Stembridge originally introduced the more general notion of an *enriched (P, ω)-partition* (for ω a labeling of P) along with its corresponding weight enumerator. His definitions specialize to ours when ω is a natural labeling. A key fact from [36] is that an enriched weight enumerator is the sum of peak functions:

$$\widetilde{K}_P = \sum_{\sigma \in \mathcal{L}(P)} \theta_{Peak(\sigma)}. \tag{5.4}$$

We state the main result of this section.

Theorem 5.1. *We have*

$$2F_{\widehat{B}(P)^*} = \widetilde{K}_{P_0}.$$

Proof. It follows from [36, Theorem 3.6 and (1.4)] that

$$\begin{aligned} \widetilde{K}_{P_0} &= \sum_{(\varepsilon,\sigma) \in \{\pm 1\}^{\times(n+1)} \times \mathcal{L}(P_0)} L_{Des(\varepsilon_0\sigma_0, \ldots, \varepsilon_n\sigma_n)} \\ &= 2 \sum_{(\varepsilon,\sigma) \in \{\pm 1\}^{\times n} \times \mathcal{L}(P)} L_{Des(0,\, \varepsilon_1\sigma_1, \ldots, \varepsilon_n\sigma_n)}. \end{aligned} \tag{5.5}$$

The last expression equals $2F_{\widehat{B}(P)^*}$ by Proposition 2.5 and the fact that, by Theorem 4.1, the induced edge-labeling of $\widehat{B}(P)^*$ is an R-labeling. $\square$

Remark 5.2. In [4, Example 7.5] it is observed that $\widetilde{K}_{P_0} = \sum_c L_{Des(c)}$, the sum being over all maximal chains in the doubled reséau $\delta J(P_0)$. This formula is essentially (5.5) and thus provides an alternate approach to proving Theorem 5.1. Yet another proof can be adapted from that of [6, Theorem 3.1]; see Remark 5.16.

5.3 *Order polynomials and chain polynomials*

Following Stembridge [36], we define the *enriched order polynomial* of P by

$$\Omega'_P(m) := \widetilde{K}_P(\underbrace{1, \ldots, 1}_{m}, 0, \ldots).$$

Alternately, $\Omega'_P(m)$ is the number of enriched P-partitions $\sigma : P \to \pm\mathbb{P}$ such that $\sigma(p) \preccurlyeq m$ for all $p \in P$. As an enriched analog of the familiar equation $Z_{J(P)}(m) = \Omega_P(m)$ relating the zeta polynomial of $J(P)$ to the order polynomial of P (see [34, page 130]), we obtain

Corollary 5.3. *We have*

$$2\, Z_{\widehat{B}(P)}(m) = \Omega'_{P_0}(m). \tag{5.6}$$

Proof. The proof is immediate from Theorem 5.1, (5.1), and the definition of Ω'. $\square$

Define the polynomials $W_P(t)$ and $W'_P(t)$ by

$$W_P(t) := \sum_{\sigma \in \mathcal{L}(P)} t^{\#Des(\sigma)+1},$$

$$W'_P(t) := \sum_{\sigma \in \mathcal{L}(P)} t^{\#Peak(\sigma)+1}.$$

The fundamental identities for generating functions of ordinary and enriched order polynomials are

$$\sum_{m \geq 0} \Omega_P(m) t^m = \frac{1}{(1-t)^{n+1}} \cdot W_P(t), \tag{5.7}$$

$$\sum_{m \geq 0} \Omega'_P(m) t^m = \frac{1}{2} \left(\frac{1+t}{1-t} \right)^{n+1} \cdot W'_P\left(\frac{4t}{(1+t)^2} \right). \tag{5.8}$$

These are due to Stanley [34, Theorem 4.5.14] and Stembridge [36, Theorem 4.1], respectively.

In 1978, Neggers [27] conjectured that the polynomial $W_P(t)$ should have only real zeros (assuming, as we have been, that P is an arbitrary naturally labeled poset on $[n]$). Stembridge in turn conjectured that $W'_P(t)$ should also have only real zeros [36]. Very recently, an extensive computer search carried out by Stembridge [37] produced counterexamples to both of these conjectures. The search was initiated after Brändén [17] discovered counterexamples to Stanley's extension of Neggers' conjecture, which predicted that the W-polynomial should have only real zeros even for posets that are not naturally labeled. See [18], [19], and [29] for references and partial results on the Neggers-Stanley Conjecture. Presently, it remains an open problem to determine whether the polynomials $W_P(t)$ and $W'_P(t)$ have unimodal coefficients, even when the labeling of P is not assumed to be natural.

One way to understand these polynomials is to relate them to certain chain polynomials. Recall that the *chain polynomial* of a graded poset Q of rank n is defined by $C_Q(t) := \sum_{i=0}^{n} c_i t^i$, where c_i is the number of chains in Q of length i from $\hat{0}$ to $\hat{1}$. By standard manipulations of generating functions, as in [34, Chapter 3, Exercise 67], one can show that

$$\sum_{m\geq 0} Z_Q(m)t^m = \frac{1}{1-t}\cdot C_Q\left(\frac{t}{1-t}\right). \tag{5.9}$$

Since $Z_{J(P)}(m) = \Omega_P(m)$, it follows that

$$W_P(t) = (1-t)^n \cdot C_{J(P)}\left(\frac{t}{1-t}\right), \tag{5.10}$$

which is well known. We state the analogous formula for $W'_P(t)$:

Proposition 5.4. *If* $u = \sqrt{1-t}$, *then*

$$W'_{P_0}(t) = 2\,u^{n+1}\,(1+u)\cdot C_{\widehat{B}(P)}\left(\frac{1-u}{2u}\right). \tag{5.11}$$

Proof. Combining (5.6), (5.8), and (5.9), we get

$$\frac{1}{4}\left(\frac{1+t}{1-t}\right)^{n+2}\cdot W'_{P_0}\left(\frac{4t}{(1+t)^2}\right) = \frac{1}{1-t}\cdot C_{\widehat{B}(P)}\left(\frac{t}{1-t}\right).$$

The proof is complete upon substituting $(1-u)/(1+u)$ for t. □

Remark 5.5. Stembridge [37] devised efficient algorithms to compute certain polynomials he denotes by Z_P and $\overline{Z}_P$ (not to be confused with zeta polynomials) and then showed how Z_P relates to W_P, and $\overline{Z}_P$ to W'_P, by a change of variables. Comparing (5.10) to [37, Proposition 2.1(a)], and (5.11) to [37, Proposition 2.1(b)], it is evident that Z_P is the chain polynomial of $J(P)$ and that $\overline{Z}_{P_0}$ is twice the chain polynomial of $\widehat{B}(P)$.

Björner and Farley [14] recently showed that the chain polynomial of a distributive lattice is "75% unimodal" (meaning the first half of the coefficients are increasing and the last quarter are decreasing) by considering the geometry of the order complex of $J(P)\backslash\{\hat{0},\hat{1}\}$. According to Theorem 4.7, the order complex of $\widehat{B}(P)\backslash\{\hat{0},\hat{1}\}$ triangulates a sphere. Thus an older result due to Björner [13] implies that the chain polynomial of $\widehat{B}(P)$ is also 75% unimodal. In a forthcoming paper with Billera and Provan, we show that the order complex of $\widehat{B}(P)\backslash\{\hat{0},\hat{1}\}$ is realizable as the boundary of a simplicial polytope. Hence, the "g-Theorem" for simplicial polytopes ensures that the *h-polynomial* $h_{\widehat{B}(P)}(t) := (1-t)^{n+1}C_{\widehat{B}(P)}(t/(1-t))/t$ is symmetric and unimodal. See [5] for further background.

It is natural to ask if these unimodality properties can be formally transferred to the polynomials $W_P(t)$ or $W'_P(t)$. In fact, there are simple examples showing this not to be the case. For instance, evaluating the right-hand side of (5.11) with the (75% unimodal) polynomial $C(t) = t + 11t^2 + 52t^3 + 118t^4 + 125t^5 + 50t^6$ in place of $C_{\widehat{B}(P)}(t)$ yields the non-unimodal polynomial $(16t + 4t^2 + 5t^3)/16$. This occurs despite the fact that the corresponding h-polynomial $(1-t)^6\ C(t/(1-t))/t = 1 + 6t + 18t^2 + 18t^3 + 6t^4 + t^5$ is symmetric and unimodal.

5.4 The **cd**-*index*

Theorem 5.1 may be used to give a combinatorial interpretation of the **cd**-index of $\widehat{B}(P)$. For a graded poset Q of rank n, define a polynomial of degree $n-1$ in the non-commuting variables $\mathbf{a}$ and $\mathbf{b}$ of degree 1 by

$$\Psi_Q := \sum_{S\subseteq[n-1]} f_S(Q)u_S,$$

where $u_S = u_1\cdots u_{n-1}$, $u_i = \mathbf{b}$ if $i \in S$ and $u_i = \mathbf{a}-\mathbf{b}$ if $i \notin S$. Fine observed and Bayer-Klapper proved that when Q is Eulerian, Ψ_Q can be written as a polynomial in the variables $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$, called the **cd**-*index* of Q [1]. For a sampling of work on the **cd**-index, see [33], [6], [7], [24], and [22]. If Γ is a cell complex such that $\widehat{P}(\Gamma)$ is Eulerian, we may refer to $\Psi_{\widehat{P}(\Gamma)}$ as the **cd**-index of Γ or $P(\Gamma)$.

To connect the **cd**-index to our work, we set up a one-to-one correspondence

$w \mapsto S_w$ between the set of **cd**-words of degree $n-1$ and $Peak_n$ given by

$$\mathbf{c}^{a_1}\mathbf{d}\mathbf{c}^{a_2}\mathbf{d}\cdots\mathbf{c}^{a_k}\mathbf{d}\mathbf{c}^{a_{k+1}} \mapsto \{\deg(\mathbf{c}^{a_1}\mathbf{d}), \deg(\mathbf{c}^{a_1}\mathbf{d}\mathbf{c}^{a_2}\mathbf{d}), \ldots, \deg(\mathbf{c}^{a_1}\mathbf{d}\cdots\mathbf{c}^{a_k}\mathbf{d})\}.$$

For fixed n, let w_S denote the **cd**-word of degree $n-1$ associated to the peak set $S \in Peak_n$. For instance, $S_{\mathbf{cddccdc}} = \{3,5,9\} \in Peak_{11}$ and $w_{\{3,5,9\}} =$ **cddccdc**. Given an Eulerian poset Q of rank n and a **cd**-word w of degree $n-1$, let $[w]$ denote the coefficient of the word w in Ψ_Q. A link between the **cd**-index and the peak algebra is provided by the identity [8, Corollary 2.2]

$$F_Q = \sum_{S \in Peak_n} \frac{[w_S]}{2^{1+\#S}}\, \theta_S. \tag{5.12}$$

This formula together with Theorem 5.1 and (5.4) yield

Theorem 5.6. *We have*

$$\Psi_{\widehat{B}(P)^*} = \sum_{\sigma \in \mathcal{L}(P_0)} 2^{\#Peak(\sigma)} w_{Peak(\sigma)}.$$

In particular, the **cd***-indices of $\widehat{B}(P)^*$ and $\widehat{B}(P)$ have non-negative coefficients.*

Note that Ψ_{Q^*} is obtained from Ψ_Q by changing every **cd**-word w to w^*, the word consisting of the letters of w in reverse order; see, e.g., [6].

Example 5.7. If P is the poset from Figure 1(a) then

$$\begin{aligned}\Psi_{\widehat{B}(P)^*} &= w_{Peak(0123)} + w_{Peak(0213)} + w_{Peak(0231)} \\ &= w_\emptyset + 2w_{\{2\}} + 2w_{\{3\}} \\ &= \mathbf{ccc} + 2\mathbf{dc} + 2\mathbf{cd}\end{aligned}$$

and

$$\Psi_{\widehat{B}(P)} = \mathbf{ccc}^* + 2\mathbf{dc}^* + 2\mathbf{cd}^* = \mathbf{ccc} + 2\mathbf{cd} + 2\mathbf{dc}.$$

Theorem 5.6 provides further evidence for Stanley's Gorenstein* conjecture [33, Conjecture 2.1], which is known to hold for face lattices of convex polytopes and oriented matroids:

Conjecture 5.8 (Stanley). *The coefficients of the* **cd***-index of a Gorenstein* poset are non-negative.*

Remark 5.9. Conjecture 5.8 has received special attention in connection with a conjecture of Charney and Davis [20] on the sign of the quantity

$$\kappa(\Gamma) := 1 - \frac{1}{2}f_0 + \frac{1}{4}f_1 - \cdots + \left(-\frac{1}{2}\right)^{d+1} f_d,$$

where f_i is the number of i-cells of the d-dimensional cell complex Γ. The Charney-Davis Conjecture predicts that $(-1)^m \kappa(\Gamma) \geq 0$ whenever Γ is a *flag complex* triangulating a $(2m-1)$-sphere. If Γ is the order complex of $Q\backslash\{\hat{0}, \hat{1}\}$, where Q is an Eulerian poset of rank $2m+1$, then $(-1)^m 2^{2m} \kappa(\Gamma)$ is the coefficient of $\mathbf{d}^m$ of the **cd**-index of Q. See [32] for additional details. For the face poset Q of a cell complex Γ, the order complex of $Q\backslash\{\hat{0}\}$ is a flag complex and is the barycentric subdivision of Γ. Thus Theorem 5.6 proves a special case of the Charney-Davis Conjecture by supplying a combinatorial interpretation of the quantity $(-1)^m \kappa(\Gamma)$ when Γ is the barycentric subdivision of a cellular sphere whose face poset is a signed Birkhoff poset.

Taking P to be an antichain in Theorem 5.6 yields [6, Proposition 8.1]:

Corollary 5.10 (Billera, Ehrenborg, and Readdy). *Let $\mathcal{C}_n$ be the face lattice of the n-dimensional cube. Let $\mathfrak{S}_n^0$ be the set of permutations of $0, 1, \ldots, n$ that start with 0. Then*

$$\Psi_{\mathcal{C}_n} = \sum_{\pi \in \mathfrak{S}_n^0} 2^{\# Peak(\pi)} w_{Peak(\pi)}.$$

On the other hand, if P is a chain then clearly $\Psi_{\widehat{B(P)}} = \mathbf{c}^n$. For arbitrary P, $\mathcal{L}(P_0)$ is a subset of $\mathfrak{S}_n^0$. Thus Theorem 5.6 and Corollary 5.10 imply

Corollary 5.11. *The* **cd**-*index of a signed Birkhoff poset of rank $n+1$ is coefficient-wise maximized by the* **cd**-*index of the n-dimensional cross-polytope and minimized by $\mathbf{c}^n$. In other words, $\Psi_{\widehat{B(P)}}$ is coefficient-wise maximized when P is an antichain minimized when P is a chain.*

Remark 5.12. In contrast to Corollary 5.11, there are the many results and conjectures giving non-trivial lower bounds for the **cd**-index over various classes of posets. For example, over the class of (face posets of) convex polytopes, the **cd**-index is minimized on the simplex [7]; over the class of oriented matroids it is minimized on the cross-polytope [6]; over the class of Cohen-Macaulay cubical posets it is conjectured to be minimized on the cube [23]; over the class of Gorenstein* lattices it is conjectured to be minimized on the simplex [33].

5.5 Comparisons with oriented matroids.

Let Γ be a cell complex whose face poset is isomorphic to $B(P)$ for some P. Let m be the number of minimal elements of P. The number of maximal cells of Γ is clearly 2^m, which equals

$$\sum_{x \in J(P)} |\mu_{J(P)}(\hat{0}, x)|, \tag{5.13}$$

where $\mu_{J(P)}$ is the Möbius function of $J(P)$. This is verified in the proof of Proposition 5.13 below.

The cell-count formula (5.13) is reminiscent of a famous result of Zaslavsky expressing the f-vector of a hyperplane arrangement in terms of its intersection lattice [38]. He showed in particular that the number of regions in a hyperplane arrangement is $\sum_{x \in L} |\mu_L(\hat{0}, x)|$, where L is the intersection lattice. This result holds more generally in the setting of oriented matroids, where the intersection lattice is replaced by the geometric lattice of flats. We refer the reader to [15] for background and references in this area. Note that whereas a signed Birkhoff poset is completely determined by its underlying distributive lattice, an oriented matroid is not necessarily determined by its geometric lattice. In this respect, Zaslavsky's formula is more surprising, and indeed more subtle, than (5.13). Bayer and Sturmfels [2] extended Zaslavsky's result by showing that the flag f-vector of an oriented matroid depends only on the associated geometric lattice. The dependency is formulated explicitly in [15, Proposition 4.6.2] in terms of the zero map, which "forgets the signs" of covectors. Using φ in place of the zero map, we have an essentially identical formula:

Proposition 5.13. *Let $A_k < A_{k-1} < \cdots < A_0 = \emptyset$ be a chain in $J(P)$. The number of chains in the preimage of c under the map $\varphi : B(P) \to J(P)$ is*

$$\#\varphi^{-1}(c) = \prod_{i=1}^{k} \sum_{\substack{B \in J(P) \\ A_i \leq B \leq A_{i-1}}} |\mu(A_i, B)|,$$

where μ is the Möbius function of $J(P)$.

Proof. Throughout this proof, $\mathsf{Min}(S)$ denotes the set of minimal elements of a subset S of $\pm P$ relative to $\leq_{\pm P}$. Thus, if $S \subseteq P$, then $\mathsf{Min}(S)$ consists of the minimal elements of S relative to $\leq_P$.

We count chains $X_1 < X_2 < \cdots < X_k$ such that $\varphi(X_i) = A_i$ for all i. Set $X_0 = \emptyset$. Assume that for some $j \geq 1$ we have chosen $X_0 < X_1 < \cdots < X_{j-1}$ such $\varphi(X_i) = A_i$ for $i = 0, \ldots, j-1$. Let $S = \{Y \in B(P) : Y > X_{j-1}$ and $\varphi(Y) = A_j\}$. It is clear that if $Y \in S$ then $\varphi(\mathsf{Min}(Y \backslash X_{j-1})) = \mathsf{Min}(A_j \backslash A_{j-1})$. Moreover, an arbitrary assignment of $+$ or $-$ signs to elements of $\mathsf{Min}(A_j \backslash A_{j-1})$ determines a set of the form $\mathsf{Min}(Y \backslash X_{j-1})$ for a unique $Y \in S$. Thus there are $\#S = 2^{\#\mathsf{Min}(A_j \backslash A_{j-1})}$ ways to complete the chain $X_1 < \cdots < X_{j-1}$ to $X_1 < \cdots < X_j$ such that $\varphi(X_j) = A_j$. Doing this for each $j = 1, \ldots, k$

yields the formula

$$\#\varphi^{-1}(c) = \prod_{i=1}^{k} 2^{\#\mathsf{Min}(A_i \backslash A_{i-1})}.$$

The proof will be complete once we show that for any $i \in [k]$,

$$\sum_{\substack{B \in J(P) \\ A_i \leq B \leq A_{i-1}}} |\mu(A_i, B)| = 2^{\#\mathsf{Min}(A_i \backslash A_{i-1})}.$$

It follows easily from well-known facts about the Möbius function of a distributive lattice (e.g., [34, Example 3.9.6]) that

$$|\mu(A_i, B)| = \begin{cases} 1 & \text{if } A_i \backslash B \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A_i \backslash B$ is an antichain if and only if $A_i \backslash B \subseteq \mathsf{Min}(A_i \backslash A_{i-1})$. But then $\#\{B \in [A_i, A_{i-1}] : A_i \backslash B \subseteq \mathsf{Min}(A_i \backslash A_{i-1})\} = 2^{\#\mathsf{Min}(A_i \backslash A_{i-1})}$, which is what we needed to prove. □

Billera, Ehrenborg, and Readdy [6] described a simple way to compute the **cd**-index of an oriented matroid in terms of the flag f-vector of the underlying geometric lattice. Their result was originally stated in terms of a linear map ω on the vector space of **ab**-polynomials. We will state their result in terms of a corresponding map on quasisymmetric functions. Let us define a linear map $\vartheta : \mathcal{Q} \to \Pi$ on the basis $\{L_S\}$ by

$$\vartheta(L_{Des(\sigma)}) = \theta_{Peak(\sigma)}$$

for any fixed $n \geq 1$ and any sequence of $\sigma = (\sigma_1, \ldots, \sigma_n)$. We set $\vartheta(1) = 1$. It is easy to see that ϑ is well-defined. Stembridge [36] introduced ϑ as a means of relating the weight enumerator of P-partitions to that of enriched P-partitions. A basic consequence of the definition of ϑ is that

$$\vartheta(K_P) = \widetilde{K}_P. \tag{5.14}$$

It is also possible to view ϑ as a specialization of a family of maps on noncommutative symmetric functions studied by Krob, Leclerc, and Thibon [26]. Many properties of these maps, such as diagonalizability, are proved in their work, and connections to the peak algebra are explained in [3].

Aguiar and Bergeron were the first to point out that ϑ is essentially the map ω defined in [6]. Based on this observation one can state [6, Theorem 3.1] as follows (see [8, Proposition 3.5]):

Theorem 5.14 (Billera, Ehrenborg, and Readdy). *Let L be the geometric* *lattice of an oriented matroid $\mathcal{O}$. Then*

$$2F_{T^*} = \vartheta(F_{L_0}),$$

where T is the face lattice of $\mathcal{O}$.

By comparison, using (5.3) and (5.14), we can restate Theorem 5.1 as follows:

Theorem 5.15. *We have*

$$2F_{\widehat{B}(P)^*} = \vartheta(F_{J(P_0)}).$$

Remark 5.16. It is possible to prove Theorem 5.15 (and hence Theorem 5.1) by adapting Billera, Ehrenborg, and Readdy's proof of [6, Theorem 3.1], with Proposition 5.13 now playing the role of [15, Proposition 4.6.2].

Theorem 5.15 summarizes the relationship between the flag enumerative invariants of a signed Birkhoff poset and its underlying distributive lattice.

In light of the previous two theorems, it would be of interest to find other natural examples of pairs of posets (Q, Q') such that Q is Eulerian, Q' is graded, and $2F_Q = \vartheta(F_{Q'})$.

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