

Flag Vectors of Polytopes
An Overview

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Definitions

CONVEX POLYTOPE:

$$P = \text{conv}\{x_1, x_2, \dots, x_n\} \subset \mathbf{R}^d$$

proper FACE:

intersection of supporting hyperplane with P

FACE LATTICE:

\emptyset , P , and proper faces, ordered by inclusion

FACE VECTOR:

$$(f_0(P), f_1(P), \dots, f_{d-1}(P))$$

The main problem

BIG PROBLEM:

Characterize the face vectors of d -dimensional convex polytopes.

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THEOREM (STEINITZ, 1906)

$(f_0, f_1, f_2) \in \mathbf{N}^3$ is the face vector of a 3-dimensional convex polytope if and only if

1. $f_0 - f_1 + f_2 = 2$ and
2. $2f_1 \geq 3f_0$ and $2f_1 \geq 3f_2$.

The g -theorem

THEOREM (Conjectured by McMullen; proved by Stanley, and Billera and Lee 1980)

Characterization of all face vectors of simplicial polytopes (all proper faces simplices)

- linear equations (Dehn-Sommerville)
- linear inequalities
- nonlinear inequalities

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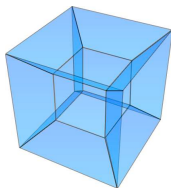
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NONSIMPLICIAL, $\dim \geq 4$?

still open

need to look further than the face vector . . .



Flag vectors

Let $S = \{s_1, s_2, \dots, s_k\} < \subseteq \{0, 1, \dots, d-1\}$.

Definition

An **S-flag of P** is a chain

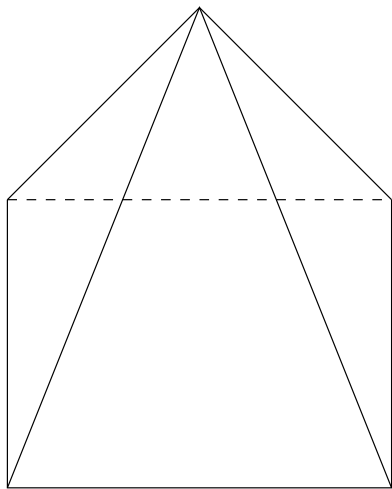
$$\emptyset \subset F_1 \subset F_2 \subset \dots \subset F_k \subset P$$

with $\dim F_i = s_i$.

$f_S(P) = \#$ of S-flags of P

$(f_S(P))_{S \subseteq \{0,1,\dots,d-1\}}$ is the **flag vector of P**.

Example



$$f_{\emptyset} = 1$$

$$f_0 = 5$$

$$f_1 = 8$$

$$f_2 = 5$$

$$f_{01} = 16$$

$$f_{02} = 16$$

$$f_{12} = 16$$

$$f_{012} = 32$$

Why Study Flag Vectors of Polytopes?

- Stanley (1970s) studied (f_S) for balanced simplicial complexes/order complexes of graded posets.
- For 3-dimensional polytopes and simplicial polytopes, for which the face vectors are characterized, the flag vector depends (linearly) on the face vector.
- For general polytopes, the flag vector reflects greater combinatorial complexity than the face vector.
- Inequalities on flag vectors project to inequalities on face vectors.
- Flag vectors relate to parameters from algebraic geometry.

Generalized Dehn-Sommerville Equations

THEOREM (B-Billera 1983)

The dimension of the linear span of flag vectors of d -polytopes is the d th Fibonacci number.

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- Kalai 1988 gives an elegant basis of polytopes.

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Independent flag numbers

- dimension 3: f_{\emptyset}, f_0, f_1
- dimension 4: $f_{\emptyset}, f_0, f_1, f_2, f_{02}$
- dimension 5: $f_{\emptyset}, f_0, f_1, f_2, f_3, f_{02}, f_{03}, f_{13}$

Inequalities on Flag Vectors

Kalai rigidity inequality (1987)

$$f_{02} - 3f_2 + f_1 - df_0 + \binom{d+1}{2} \geq 0$$

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4-dimensional polytopes (B 1987)

$$f_0 \geq 5$$

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$$f_{02} - 3f_2 \geq 0$$

$$f_{02} - 3f_1 \geq 0$$

$$f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0$$

$$6f_1 - 6f_0 - f_{02} \geq 0$$

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Compare: face
vectors of
4-polytopes by
Barnette,
Grünbaum, Reay,
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Not known to be best possible.

No further linear inequalities for $d = 4$ since 1987.

Toric h -vector (Stanley 1987)

$(h_0, h_1, h_2, \dots, h_d)$

middle perversity intersection homology Betti numbers

$(h_0, h_1, h_2, \dots, h_d)$ depends linearly on flag vector, but not vice versa

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$$1 = h_0 \leq h_1 \leq h_2 \leq \dots \leq h_{\lfloor d/2 \rfloor}$$

from algebraic geometry (for rational polytopes)

imply linear inequalities on flag vectors

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Karu 2004 broke dependence on algebraic geometry, extending results to nonrational polytopes.

cd-index

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4-simplex: $cccc + 3dcc + 5cdc + 3ccd + 4dd$

4-cube: $cccc + 14dcc + 16cdc + 6ccd + 20dd$

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Fine conjecture

Each coefficient in the cd -index of a polytope is ≥ 0 .

Proved by Stanley 1994 (for S -shellable spheres), by Ehrenborg and Karu (2007) for Gorenstein* posets

Strengthened by Billera and Ehrenborg 2000: The d -simplex minimizes each coefficient among d -polytopes (and Ehrenborg and Karu (2007) for Gorenstein* posets)

Linear inequalities on flag vectors

Basic linear inequalities come from

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How to analyze the resulting set of inequalities?

- Which are redundant?
- Which give facets of the closed convex cone of flag vectors?

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Some answers by ...

- Ehrenborg 2005
- Stenson 2004, 2005

Kalai's convolution

Definition

$$S \subseteq \{0, 1, \dots, d-1\} \quad T \subseteq \{0, 1, \dots, e-1\}$$

$$\begin{aligned} f_S * f_T(P) &= \sum_{\dim F=d} f_S(F) f_T(P/F) \\ &= f_{S \cup \{d\} \cup (T+(d+1))}(P) \end{aligned}$$

flag number of $(d + e + 1)$ -dimensional polytope
(extend linearly for linear forms in flag numbers)

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Convolutions produce inequalities

If m_d is a nonnegative linear form in f_S , $S \subseteq \{0, 1, \dots, d-1\}$,
and n_e is a nonnegative linear form in f_T ,
 $T \subseteq \{0, 1, \dots, e-1\}$,
then $m_d * n_e \geq 0$ for $(d+e+1)$ -polytopes.

Ehrenborg's lifting

Example

For a cd -word w , and a convex polytope P , write $[w]$ for the coefficient of w in the cd -index of P .

Kalai's rigidity inequality for 4-polytopes

$$f_{02}(P) - 3f_2(P) + f_1(P) - 4f_0(P) + 10 \geq 0$$

can be written in terms of the cd -index, as

$$[dd] - [ccd] - [dcc] + 2[cccc] \geq 0.$$

Ehrenborg lifting then gives:

For every cd -words u and v where u does not end in c and $\deg u + \deg v = n$, for every $(n + 4)$ -dimensional polytope

$$[uddv] - [uccdv] - [udccv] + 2[ucccv] \geq 0.$$

Flag vectors of 4-dimensional polytopes

toric h -vector and cd -index don't give new linear inequalities for 4-polytopes

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Ziegler

focus on fatness/complexity gives better understanding, suggests directions and constructions

Fatness

$$F(P) = \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$$

$$5/2 \leq F(P)$$

Is there an upper bound for $F(P)$?

Largest known $F(P) < 9$

Recent results on 4-polytopes

Paffenholz and Werner 2006

construction of infinite family of 4-polytopes that are 2-simplicial, 2-simple, and elementary
gives extreme ray of cone of flag vectors

Definitions

A polytope is **2-simplicial** if every 2-face is a triangle.

A polytope is **2-simple** if every edge is contained in exactly 3 facets.

A polytope P is **elementary** if

$$f_{02}(P) - 3f_2(P) + f_1(P) - 4f_0(P) + 10 = 0$$

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Other examples of 2-simplicial, 2-simple polytopes (Eppstein, Kuperberg, Paffenholz, Ziegler)

Recall inequalities on 4-dimensional polytopes

$$f_0 \geq 5$$

$$f_3 \geq 5$$

$$f_{02} - 3f_2 \geq 0 \quad (\text{equality for 2-simplicial})$$

$$f_{02} - 3f_1 \geq 0 \quad (\text{equality for 2-simple})$$

$$f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \quad (\text{equality for elementary})$$

$$6f_1 - 6f_0 - f_{02} \geq 0$$

More new results on k -polytopes

Ling 2006

new nonlinear inequalities for flag vectors

$$(k-1)f_{02} - \binom{k+1}{2}f_2 + f_1 \leq \binom{f_0}{2}$$

$$2(k-1)f_{02} - k(k+1)f_2 + (k^2 - 3k + 4)f_1 - k(k-3)f_0 \leq 4 \binom{f_0}{2}$$

Difficulty

We do not know how to generate **random combinatorial types** of polytopes.

Note that the convex hull of a random set of points in \mathbf{R}^d is a simplicial polytope.

This makes it difficult to test conjectures, and even to come up with conjectures.

Further work—specialization

Special classes of polytopes

- cubical (Adin, Babson, G. Blind, R. Blind, Chan, Hetyei, Jockusch, Joswig, Liu, Ziegler)
- k -simplicial, h -simple (Kalai, Paffenholz, Stenson, Werner, Ziegler)
- polytopes with symmetry (A'Campo-Neuen, Adin, Björner, Jorge, Novik, Stanley)
- zonotopes and geometric lattices (B, Billera, Ehrenborg, Kung, Nyman, Readdy, Stenson, Sturmfels, Swartz)
- 0/1 polytopes (Aichholzer, Bárány, Gatzouras, Giannopoulos, Kaibel, Markoulakis, Pór, Ziegler)
- flag complexes (Athanasiadis, Branden, Nevo)
- cyclic-like polytopes (B, Bisztriczky, Dinh, Smilansky)

Cyclic-like polytopes

Generalizations of the simplex

multiplex

braxtope

Gale polytopes

facets satisfy “Gale’s evenness condition”

Generalizations of cyclic polytopes

simplicial and Gale cyclic polytope

multiplicial and Gale ordinary polytope

braxial and Gale periodically cyclic
Gale polytope

Further work—generalization

More general classes of partially ordered sets

- general graded posets (Billera, Hetyei, Liu)
- Eulerian posets (B, Billera, Chen, Ehrenborg, Hetyei, Jojić, Lau, Readdy, Reading, Stanley)
- Eulerian manifolds (Björner, Charney, Chen, Davis, Hersh, Kalai, Novik, Sparla, Yan)
- Gorenstein* lattices (Billera, Ehrenborg, Karu, Masuda, Murai, Readdy, Reading, Stanley)

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Connections with other mathematical structures

- toric varieties (Bressler, Bukhshtaber, Karu, Leung, Lunts, Panov, Reiner, Stanley)
- coalgebras (Ehrenborg and Readdy)
- Hopf algebra of quasisymmetric functions (Aguiar, N. Bergeron, Billera, Hsiao, Sottile, van Willigenburg)

The end

THANK YOU!

<http://www.math.ku.edu/~bayer>