Abelian vs. nonabelian equivariant K-theory

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Notation and Definitions

Let G be a compact Lie group (with $\pi_1(G)$ torsion-free).

Let $T \subset G$ be a maximal torus, with Weyl group $W = N_T/T$.

Let X be a compact G-space. By restriction X is also a T-space.

Definition (Atiyah-Segal)**.** The *G-equivariant K-theory* of a compact G-space X is given by

$$K_G(X) := K\big(\mathrm{Vect}_G(X)\big),$$

the Grothendieck ring of virtual G-equivariant complex vector bundles over X.

Note. This is *not* the same as the Borel equivariant K-theory, $K(EG \times_G X)$, which is actually a completion of $K_G(X)$.

Other Definitions

We can extend our definition to non-compact G-spaces by taking

$$K_G(X) := \big[X, \operatorname{Fred}(\mathcal{H}_G)\big]_G,$$

the space of homotopy classes of G-equivariant maps from X to Fredholm operators on a complex, separable Hilbert space $\mathcal{H}_G$ containing each finite dimensional irreducible representation of G with infinite multiplicity.

Alternatively, one can define equivariant K-theory in terms of the KK-theory of C^*-algebras.

Both of these definitions can be further extended to allow for twisted equivariant K-theory, letting us generalize our results.

Our Problem

We want to compute $K_G(X)$ in terms of $K_T(X)$.

Often, it is significantly simpler to compute equivariant cohomology theories with respect to an abelian group than a non-abelian group.

Our motivation from equivariant symplectic geometry:

- Martin's theorem computes the rational cohomology of the non-abelian symplectic quotient $H^*(X/\!/G;\mathbb{Q})$ in terms of the abelian symplectic quotient $H^*(X/\!/T;\mathbb{Q})$.
- Harada and I can compute the K-theory of abelian symplectic quotients $K^*(X/\!/T)$, but we cannot yet compute $K^*(X/\!/G)$ for non-abelian symplectic quotients.

Precedents

In the nicest cases, the G-equivariant cohomology is precisely the Weyl invariant part of the T-equivariant cohomology.

For Borel-equivariant rational cohomology,

$$H_G^*(X;\mathbb{Q}) \cong H_T^*(X;\mathbb{Q})^W.$$

For representation rings (i.e., equivariant K-theory of a point),

$$R(G) \cong R(T)^W.$$

However, this is *not* true in general for Borel equivariant integral cohomology (see Holm and Sjamaar) or equivariant K-theory.

Counterexample

Theorem (McLeod). *If $K^*_T(X)$ is a free module over $R(T)$, then*

$$K^*_G(X) \cong K^*_T(X)^W.$$

However, the free module requirement is unduly restrictive.

For example, let $X = \mathrm{SU}(2) \times \mathbb{R}P^2$ with $G = \mathrm{SU}(2)$ acting freely on the $\mathrm{SU}(2)$ factor and trivially on the $\mathbb{R}P^2$ factor. We have

$$K_{\mathrm{SU}(2)}\big(\mathrm{SU}(2) \times \mathbb{R}P^2\big) \cong K\big(\mathbb{R}P^2\big) \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$

while

$$K_{\mathrm{U}(1)}\big(\mathrm{SU}(2) \times \mathbb{R}P^2\big) \cong K\big(S^2 \times \mathbb{R}P^2\big) \cong (\mathbb{Z} \oplus \mathbb{Z}H) \otimes (\mathbb{Z} \oplus \mathbb{Z}_2),$$

with $W = \mathbb{Z}_2$ taking the Hopf bundle $H \mapsto H^{-1} = 2 - H$. Thus

$$K_{\mathrm{U}(1)}\big(\mathrm{SU}(2) \times \mathbb{R}P^2\big)^W = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

From G to T

Theorem (Hodgkin, Snaith, McLeod)**.**

$$K_T^*(X) = K_G^*(X) \otimes_{R(G)} R(T).$$

Proof. This follows from the Künneth spectral sequence for equivariant K-theory, which constructs $K_G^*(X \times Y)$ starting with $K_G^*(X) \otimes_{R(G)} K_G^*(Y)$ and higher $R(G)$-torsion at the E_2-stage. Here, we take $Y = G/T$, and we have

$$\begin{aligned} K_T^*(X) &\cong K_G^*(X \times G/T) \\ &\cong K_G^*(X) \otimes_{R(G)} K_G^*(G/T) \\ &\cong K_G^*(X) \otimes_{R(G)} R(T), \end{aligned}$$

and since Pittie showed that $R(T)$ is a free $R(G)$-module, the higher torsion vanishes and the spectral sequence collapses. □

From T to G

Since $R(T)$ is a free $R(G)$-module, the algebra

$$\mathcal{E} = \operatorname{End}_{R(G)} R(T)$$

is a matrix algebra, and Morita equivalence tells us that for any $R(G)$-module A, we have

$$A \cong \operatorname{Hom}_{\mathcal{E}}\big(\, R(T),\, A \otimes_{R(G)} R(T)\,\big).$$

Furthermore, if B is an $\mathcal{E}$-module, then we have

$$\operatorname{Hom}_{\mathcal{E}}\big(R(T), B\big) \cong B^{\mathcal{I}},$$

where

$$\mathcal{I} = \{\Delta \in \mathcal{E} \mid \Delta(1) = 0\}$$

is the augmentation left ideal in $\mathcal{E}$.

Our Theorem

Theorem (Harada, L., Sjamaar)**.**

$$K_G^*(X) \cong K_T^*(X)^{\mathcal{I}},$$

where

$$\mathcal{I} = \{\Delta \in \operatorname{End}_{R(G)} R(T) \mid \Delta(1) = 0\}.$$

Note that the Weyl group W acts on $R(T)$ and fixes $R(G)$, so the group ring $\mathbb{Z}[W]$ is contained in the algebra $\mathcal{E} = \operatorname{End}_{R(G)} R(T)$. Weyl invariant elements are invariant under the augmentation ideal of $\mathbb{Z}[W]$, which is contained in the augmentation ideal $\mathcal{I}$ of $\mathcal{E}$, so

$$K_T^*(X)^{\mathcal{I}} \subset K_T^*(X)^W.$$

In light of our counterexample, this inclusion may be strict.

How $\operatorname{End}_{R(G)} R(T)$ acts on $K_T^*(X)$

For any homogeneous differential operator D on G/T, the map

$$[U] \mapsto \chi\big(\operatorname{Index}_G D \otimes (G \times_T U)\big)$$

gives an element of $\operatorname{End}_{R(G)} R(T)$ called a *push-pull operator.*

Lemma. *The push-pull operators for all homogeneous differential operators D on G/T, together with $R(T)$, generate* $\operatorname{End}_{R(G)} R(T)$.

If X is a G-space, consider the fiber bundle of G-spaces

$$G/T \longrightarrow X \times G/T \longrightarrow X,$$

where G acts on the product space via the diagonal action.

Each push-pull operator on the fiber G/T extends to a push-pull operator on $X \times G/T$ parametrized by X, giving a homomorphism

$$K_T(X) \cong K_G(X \times G/T) \xrightarrow{\operatorname{Index}_G} K_G(X) \xrightarrow{\text{res}} K_T(X).$$

Revisiting the Counterexample

For example, for $G = \mathrm{SU}(2)$ and $T = \mathrm{U}(1)$, the homogeneous Dirac operator $\not\partial$ on $\mathrm{SU}(2)/\,\mathrm{U}(1) \cong S^2$ with values in the bundle induced by the representation $\mathbb{C}_\mu$ with weight μ has SU(2)-index:

$$\mathrm{Index}_{\mathrm{SU}(2)}\, \not\partial \otimes \mathbb{C}_\mu = \pm V_{\pm\mu-1} \text{ (or 0 if } \mu = 0).$$

Taking the character, the associated push-pull operator is

$$\not\partial : \mathbb{C}_\mu \mapsto \frac{\mathbb{C}_\mu - \mathbb{C}_{-\mu}}{\mathbb{C}_1 - \mathbb{C}_{-1}},$$

which generates the augmentation ideal $\mathcal{I} \subset \mathrm{End}_{R(\mathrm{SU}(2))}\, R(\mathrm{U}(1))$.

In our counterexample, the Hopf bundle H corresponds to $\mathbb{C}_1$, and

$$\begin{aligned}(1-W) &: H \mapsto H - H^{-1} = H - (2 - H) = 2\,(H-1),\\ \not\partial &: H \mapsto 1,\end{aligned}$$

so H is annihilated by $1 - W$, but not $\not\partial$, when there is 2-torsion.

The General Case

Using the Borel-Weil-Bott theorem and Weyl Character Formula, the general push-pull operator for the Dirac operator $\not\partial$ on G/T is:

$$\mathbb{C}_\mu \mapsto \frac{\sum_{w\in W}(-1)^w \mathbb{C}_{w\mu}}{\sum_{w\in W}(-1)^w \mathbb{C}_{w\rho}},$$

where ρ is the half-sum of the positive roots. The numerator is in the group ring $\mathbb{Z}[W]$, while the Weyl denominator is the Euler class

$$e_G(G/T) \in K_G(G/T) \cong R(T).$$

Thus, we have the chain of inclusions

$$\mathbb{Z}[W] \subset \operatorname{End}_{R(G)} R(T) \subset \frac{\mathbb{Z}[W]}{e_G(G/T)},$$

and when these three spaces are equal, the Weyl invariants coincide with the $\mathcal{I}$-invariants.

When Weyl Invariants are Enough

By analyzing the structure of $\operatorname{End}_{R(G)} R(T)$, its modules, and the action of the Euler class $e_G(G/T)$, we can extends the conditions under which McLeod's Theorem hold, i.e., where the G-equivariant K-theory is the Weyl invariant part of the T-equivariant K-theory.

Corollary. *We have an isomorphism* $K_G^*(X) \cong K_T^*(X)^W$ *if*

- $K_T^*(X)$ *is a free* $R(T)$*-module,*
- *the restriction to the fixed points* $K_T^*(X) \to K_T^*(X^T)$ *is injective,*
- X *is a compact Hamiltonian* G*-manifold, or*
- X *is a nonsingular complex projective variety on which* G *acts by linear transformations.*

The Maximal Rank Case

Using the Gross, Kostant, Ramond, Sternberg formula, we can replace T with maximal rank H, and the $\not\partial$ push-pull becomes

$$U_\mu \mapsto \frac{\sum_{c \in W_G/W_H} (-1)^c \, U_{c\mu}}{\sum_{c \in W_G/W_H} (-1)^c \, U_{c\rho_G}},$$

where the c are specific representatives of the cosets in W_G/W_H.

Furthermore, the rest of the theory generalizes, giving us:

Lemma. *If G/H is* Spin$^{\mathrm{c}}$, *then the push-pull operators for all homogeneous differential operators D on G/H, together with $R(H)$, generate* $\operatorname{End}_{R(G)} R(H)$.

Theorem. *If G/H is* Spin$^{\mathrm{c}}$, *then $K^*_G(X) \cong K^*_H(X)^{\mathcal{I}_H}$, where*

$$\mathcal{I}_H = \{\Delta \in \operatorname{End}_{R(G)} R(H) \mid \Delta(1) = 0\}.$$