Equivariant Formality in K-theory

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Goal: (Joint work with Megumi Harada, McMaster University)
Compute the (integral) K-theory of symplectic quotients $M /\!/ G$.

Research Program:

1. `arXiv:math.SG/0503609`
   Atiyah-Bott Lemma: makes Morse theory equivariantly perfect.
   Kirwan surjectivity: $\kappa : K_G^*(M) \to K^*(M /\!/ G)$ is surjective.
   Equivariant formality: $K^*(M) \cong K_G^*(M) \otimes_{R(G)} \mathbb{Z}$ as rings.

2. `arXiv:math.SG/0601294`
   Martin's theorem: $K^*(M /\!/ G) \cong \dfrac{K^*(M /\!/ T)^W}{\operatorname{Ann} e_G(G/T)}$.

3. `arXiv:math.SG/0612660`
   Kirwan Injectivity: $i^* : K_T^*(M) \to K_T^*(M^T)$ is injective.
   GKM Theorem: Explicit computation of $\operatorname{im} i^*$.
   Kernel computation: Explicit computation of $\ker \kappa$.
   Example: K-theory of smooth compact toric varieties.

Equivariant Formality

For rational cohomology, a G-space M is *equivariantly formal* iff

$$H_G^*(M;\mathbb{Q}) \cong H^*(M;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(BG;\mathbb{Q})$$

as an isomorphism of modules over $H^*(BG;\mathbb{Q}) \cong H_G^*(\mathrm{pt};\mathbb{Q})$.

More generally, for any Borel equivariant cohomology theory

$$H_G^*(M) := H^*(EG \times_G M),$$

a G-space M is equivariantly formal iff the Leray-Serre spectral sequence for the fibration $M \to EG \times_G M \to BG$ collapses at E_2. Without torsion, this fibration "acts like a product in cohomology".

Equivariant K-theory using G-bundles is *not* Borel equivariant. We say a G-space M is *weakly equivariantly formal* for K-theory iff

$$K^*(M) \cong K_G^*(M) \otimes_{R(G)} \mathbb{Z},$$

as an isomorphism of *rings*, where $R(G) \cong K_G^*(\mathrm{pt})$.

Künneth Theorem

Recall the Künneth theorem for a cohomology theory $H^*(\cdot)$:

$$H^*(X) \otimes_{H^*(\mathrm{pt})} H^*(Y) \to H^*(X \times Y)$$

is an isomorphism up to $H^*(\mathrm{pt})$-torsion, via a LES.

In our case, we work with $K^*_G(\cdot)$. Taking

$$X = M \text{ and } Y = G,$$

we have a Künneth spectral sequence with edge homomorphism

$$K^*_G(M) \otimes_{R(G)} K^*_G(G) \to K^*_G(M \times G)$$
$$K^*_G(M) \otimes_{R(G)} \mathbb{Z} \to K^*(M).$$

This map is an isomorphism, i.e., M is weakly equivariantly formal, iff the higher torsion groups (from homological algebra) vanish:

$$\mathrm{Tor}^p_{R(G)}\big(K^*_G(M), \mathbb{Z}\big) = 0 \text{ for } p \neq 0.$$

The Torus Two-Step

Theorem. *If G is a compact Lie group with $\pi_1(G)$ torsion-free, compact Hamiltonian G-spaces are all weakly equivariantly formal.*

To prove this, we follow the following familiar two-step procedure:

Step 1. *Reduce from a general compact G to a maximal torus T.*

By the T-equivariant Künneth spectral sequence for M and G/T

$$K_G^*(M \times G/T) \cong K_G^*(M) \otimes_{R(G)} K_G^*(G/T)$$
$$K_T^*(M) \cong K_G^*(M) \otimes_{R(G)} R(T)$$

since $R(T)$ is a free $R(G)$-module (so its higher torsion vanishes).

It follows that a G-space M is weakly equivariantly formal for G iff it is weakly equivariantly formal for T (for $\pi_1(G)$ torsion-free).

Step 2. *Prove that compact Hamiltonian G-spaces are all weakly equivariantly formal via Morse theory and the Atiyah-Bott lemma.*

Hamiltonian T-spaces

A torus T is generated by a generic infinitesimal element $\xi \in \mathfrak{t}$. The moment map component $\mu^\xi = \langle \mu, \xi \rangle$ then has critical set

$$\mathrm{Crit}\, \mu^\xi = M^T,$$

and it follows from the Atiyah-Bott lemma that this moment map component μ^ξ is a T-equivariantly perfect Morse function.

We can therefore build $K_T^*(M)$ from the T-equivariant K-theory of the connected components $\{C_i\}$ of the T-fixed point set in M,

$$K_T^*(C_i) \cong K^*(C_i) \otimes_{\mathbb{Z}} R(T),$$

via a series of group extensions (SES's without degree shifts).

However, the higher torsion of these free $R(T)$-modules vanishes:

$$\mathrm{Tor}^p_{R(G)}\big(K_T^*(C_i), \mathbb{Z}\big) = 0 \text{ for } p \neq 0,$$

and this vanishing property is preserved under SES's. $\square$

A different sort of surjectivity

Considering the composition

$$K_G^*(M) \to K_G^*(M) \otimes_{R(G)} \mathbb{Z} \to K^*(M),$$

we obtain for a Hamiltonian G-space M with $\pi_1(G)$ torsion-free:

Corollary. *The forgetful map $K_G^*(M) \to K^*(M)$ is surjective, so every complex bundle M admits a stable lift of the G-action.*

We can remove *stable* by restricting to line bundles. The maps

$$\det : K^*(M) \to \mathrm{Pic}(M)$$
$$\det_G : K_G^*(M) \to \mathrm{Pic}_G(M)$$

are both retractions, so we obtain:

Corollary. *The forgetful map $\mathrm{Pic}_G(M) \to \mathrm{Pic}(M)$ is surjective, so every complex line bundle over M admits a lift of the G-action, and a unique lift if G is semi-simple.*