Super Poincaré Algebra

Let \( V = \mathbb{R}^{1,n-1} \) be Minkowski space. The Poincaré group \( P = \text{Spin}(V) \times V \) is the space of isometries of \( V \). We want to extend \( P \) to a super group. At the Lie algebra level, we have
\[
P = \text{spin}(V) \oplus V.
\]
To extend it, we need an odd part \( P_1 \), equipped with a \( P_0 \)-action and a symmetric bilinear map \( P_1 \otimes P_1 \rightarrow P_0 \) which is equivariant w.r.t. the \( P_0 \) action.

**Theorem.** The Clifford action \( V \rightarrow \text{End}(S) \) gives rise to a \( \text{Spin}(V) \) equivalent symmetric bilinear map \( S^* \otimes S^* \rightarrow V \).

*(Note. This does not hold for all signatures)*

**Ex. \( n=1 \)** In this case, \( \text{spin}(V) \) is trivial, and \( P = V \) is generated by \( H = \partial/\partial t \). Here, \( \text{Cl}(V) = \mathbb{R} \oplus \mathbb{R} \) and \( S = \mathbb{R} \), with \( c(H) = \pm 1 \).

The spinors are then generated by \( \Omega \) with \( \Omega^2 = \pm H \). So we have a square root of the Hamiltonian. As an example of this system, take
\[
\Omega = d + d^* \quad \text{(Dirac)}\]
\[
H = \Omega^2 = dd^* + d^*d \quad \text{(Laplacian)}
\]
Ex: \( n=2 \). Here \( V \) is generated by \( H \equiv \partial_t \), \( P \equiv \partial_x \) with \( |H|^2 = +1 \), \( |P|^2 = -1 \), and \( \text{spin}(V) \) is one dimensional with generator \( M = (0,1) \) satisfying 

Consider the light cone vectors \( W_+ = H + P, \quad W_- = H - P. \)

We have \( |W_+|^2 = |W_-|^2 = 0 \) and \( \langle W_+, W_- \rangle = 2 \).
Then \( [M, W_+] = W_+ \) and \( [M, W_-] = -W_- \), confirming that \( V \) has spin 1 w.r.t. \( \text{spin}(V) \).

Writing \( V = (W_+ \mathbb{R}) \oplus (W_- \mathbb{R}) \) and noting that \( W_\pm \mathbb{R} \) are maximal isotropic subspaces of \( V \), we can construct \( S = \Lambda^*(W_+ \mathbb{R}) \cong \mathbb{R}^2 \) with Clifford action 
\[ c(W_+) = e^{\frac{i}{2} W_+}, \quad c(W_-) = \frac{1}{\sqrt{2}} (W_-). \]
We thus have the Clifford algebra \( Cl(V) \cong \mathbb{R}^2 \).

In this notation, we have 
\[ c(H) = e^{\frac{i}{2} W_+} + i e^{\frac{i}{2} W_-} \]
\[ c(P) = e^{\frac{i}{2} W_+} - i e^{\frac{i}{2} W_-}, \]
and the Clifford action of \( M \) is 
\[ c(M) = -\frac{1}{2} c(H) c(P) \]
\[ = \frac{i}{\sqrt{2}} \left[ \begin{array}{c} e(W_+), \ i(W_-) \end{array} \right] \]
\[ = \left\{ \begin{array}{ll}
-\frac{1}{2} & \text{on } 1: \mathbb{R} \cong S^- \rightsquigarrow \text{spin } \frac{1}{2}
\end{array} \right. \]
\[ + \frac{1}{2} & \text{on } W_+: \mathbb{R} \cong S^+ \leftarrow \text{rep } S \]

As representations of \( \text{spin}(V) \), we have 
\( (S^+)^* = S^- \) and \( (S^-)^* = S^+ \).
We recall that for any vector space $V$, we have

$$\Lambda^k(V)^* \cong \Lambda^k(V^*) \cong \Lambda^k(V) \otimes \det(V^*).$$

In our case, we have $S^* \cong S \otimes W \otimes \mathbb{R}$, which gives us an inner product on $S$:

$$(s_1, s_2) = \left( \frac{1}{\sqrt{2}} W \right) (s_1^\top s_2)$$

$$\Rightarrow (1, W^+) = \frac{1}{\sqrt{2}}$$

Putting $Q_- = 1 \in S^-$, $Q_+ = \frac{1}{2} W^+ \in S^+$, we have $c(H) Q_- = Q_+$, $c(H) Q_+ = Q_-$,

$c(P) Q_- = Q_+$, $c(P) Q_+ = -Q_-$

with $\{Q_-, Q_+\} = \frac{1}{2}$, $|Q_-|^2 = |Q_+|^2 = 0$. The bilinear pairing $S^* \otimes S^* \to V$ is given by

$$s_1^* \otimes s_2^* \mapsto \sum_i V_i \langle c(V_i^*) s_1, s_2 \rangle,$$

which in our case gives

$$Q_+ \otimes Q_+ \mapsto H \cdot \epsilon^+ P \cdot \epsilon^+ = \frac{1}{2} W^+$$

$$Q_- \otimes Q_- \mapsto H \cdot \epsilon^- P \cdot \epsilon^- = \frac{1}{2} W^-$$

$$Q_+ \otimes Q_- \mapsto 0$$

or in other words $Q_+^2 = \frac{1}{2} W^+$, $Q_-^2 = \frac{1}{2} W^-$.

This arises naturally if we have a circle action on a manifold $M$ generated by a vector field $X$.

Putting $d_u = d + u \frac{i}{2} i(X)$, we have $d_{\rho u}^2 = -d_{\rho u}^2 = \frac{u}{2} L_x$

Set $Q_+ = \frac{i}{2} d_u + \frac{i}{2} d_u^*$, $Q_- = \frac{i}{2} d_u - \frac{i}{2} d_u^*$.

Then $Q_+^2 = H + P$, $Q_-^2 = H - P$, $iQ_+ Q_2 \neq 0$.
For \( n > 2 \), we have the identity
\[
Cl(\mathbb{R}^{1,n-1}) \cong Cl(\mathbb{R}^{1,1}) \otimes Cl(\mathbb{R}^{0,n-2})
\]
i.e., you can factor out the light cone. The spin representation then becomes
\[
S_{1,n-1} \cong S_{1,1} \otimes S_{0,n-2}
\]
Letting \( \mathbb{Q}^{\pm} \) be our basis for \( S_{1,1} \), we have
\[
S_{1,n-1} \cong \mathbb{Q}^{+} S_{0,n-2} \oplus \mathbb{Q}^{-} S_{0,n-2}
\]
with \( W^\pm, P_3, \ldots, P_n \) acting by
\[
c(W^\pm) = c(W^\pm) \otimes 1, \\
c(P_i) = (-1)^F \otimes c(P_i)
\]
Now, on \( S_{0,n-2} \) we have a positive definite inner product satisfying
\[
\langle P s_1, P s_2 \rangle = \langle s_1, s_2 \rangle \quad \text{for } s_1, s_2 \in S_{0,n-2}
\]
and any \( P \in \mathbb{R}^{3,n-2} \). (Just start with any inner product and average over the Clifford group, i.e., the finite group generated by an orthonormal basis \( \{ P_i \} \).) It then follows that
\[
\langle P s_1, s_2 \rangle = -(\langle s_1, P s_2 \rangle).
\]
Combining this with our metric \( \langle \Omega^+, \Omega^- \rangle = \frac{1}{2}, \quad 10 \pm 1^2 = 0 \) on \( \mathbb{R}^{1,1} \), we obtain a metric on \( \mathbb{R}^{1,n-1} \). We have
\[
\langle c(W^\pm) \rangle