BIRS 06FRG313: OFF-SHELL SUPERSYMMETRY VIA GRAPH THEORY AND SUPERSPACE

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Contents

1. Filtered Clifford supermodules

For the purposes of these notes, an off-shell representation of $1D$ supersymmetry is given by a finite dimensional $\mathbb{Z}$-graded vector space V of basic fields. The $\mathbb{Z}$-grading corresponds to engineering dimension, up to a factor of two. The degrees of freedom for the corresponding off-shell representation are given formally by $V, \partial_t V, \partial_t^2 V, \ldots,$ where the operator ∂_t has degree 2. These can be viewed as Taylor coefficients or the components of the D-module $V[\partial_t]$ (which Kevin will discuss tomorrow). Acting on our off-shell representation are the supersymmetry generators Q_i, which are operators of degree 1 satisfing the anti-commutation relation

$$Q_i Q_j + Q_j Q_i = 2\,\delta_{ij} \partial_t$$

(up to factors of i).

Modulo the identification

$$v \sim \partial_t v \sim \partial_t^2 v \sim \ldots,$$

the off-shell representation reduces to just the $\mathbb{Z}$-graded vector space V of fields, and the supersymmetry generators Q_i descend to the Clifford generators γ_i for the Clifford algebra $\mathrm{Cl}(N)$, satisfying

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\,\delta_{ij}. \tag{1.1}$$

Note that the Clifford generators γ_i interchange even and odd degree components of V, but they no longer strictly increase the integral degree by 1, as the supersymmetry generators Q_i did. So, the Clifford module structure is only $\mathbb{Z}_2$-graded (i.e., it is a supermodule), but not $\mathbb{Z}$-graded. On the other hand, defining an increasing filtration on V by

$$F_p V = \bigoplus\nolimits_{k \le p} V_p,$$

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we find that the Clifford action satisfies the identity

$$\gamma_i(F_p V) \subset F_{p+1} V. \tag{1.2}$$

Such a structure is called a *filtered Clifford (super)module.* In other words, we have

$$\gamma_i(V_p) \subset V_{p+1} \oplus V_{p-1} \oplus V_{p-3} \oplus \dots .$$

(I conjecture that we can stop at V_{p-1}.)

Going in the other direction, given a Clifford supermodule V, a filtration is a chain of subspaces

$$F_0 V \subset F_1 V \subset \cdots \subset F_{\max} V = V. \tag{1.3}$$

(This is actually called a finite, nonnegative, ascending filtration.) There is an additional condition concerning compatibility with the $\mathbb{Z}_2$-grading on V that I will not mention. Given a filtration, one can construct the associated grading:

$$\operatorname{Gr}_p V = F_p V \,/\, F_{p-1} V.$$

Since Clifford (super)modules have natural inner products (check this and state the conditions precisely), we can pass back and forth between the filtration and its associated grading without any loss of information, taking

$$F_p V \cong \bigoplus\nolimits_{k \leq p} \operatorname{Gr}_p V.$$

Given a filtered Clifford supermodule V, the associated graded vector space $\operatorname{Gr} V$ can be viewed as a collection of fields, and the Clifford generators γ_i lift to give supersymmetry generators Q_i on the corresponding off-shell representation.

In our "Filtered Clifford supermodules" paper, I proved that these constructions provide a one-to-one correspondence between engineerable off-shell representations of $1D$ supersymmetry and filtered Clifford supermodules.

In order to classify filtered Clifford supermodules, one considers $\mathbb{Z}_2$-graded flags (1.3) which satisfy the Clifford filtration condition (1.2). (Actually, due to the Clifford relations (1.1), the Clifford filtration condition (1.2) implies that the subspaces $F_p V$ form an ascending flag.) In topology and algebraic geometry, one characterizes flags by the ascending list of *filtration dimensions*

$$\dim F_0 V - \dim F_1 V - \cdots - \dim F_{\max} V.$$

This does not completely classify flags, as there is a smooth manifold of possible flags for each list of filtration dimensions. Since off-shell representations are graded by engineering dimension, it makes sense from the physics point of view to instead use the corresponding list of *graded dimensions*

$$\dim \operatorname{Gr}_0 V - \dim \operatorname{Gr}_1 V - \cdots - \dim \operatorname{Gr}_{\max} V,$$

telling us how many new fields are introduced are at each engineering degree. Toppan *et al.* mistakenly believe that the list of graded dimensions is sufficient to completely classify off-shell representations. We give a counterexample below. In light of the "Hanging Gardens Theorem", a better way to classify off-shell representations is by counting sources in the

Adinkra. In terms of filtered Clifford supermodules, this corresponds to giving the list of *source dimensions*

$$\cdots - \dim F_pV \,/\, \gamma_\bullet(F_{p-1}V) - \ldots .$$

This counts the number of fields which are not obtained by applying a supersymmetry generator to a field of lower degree. I conjecture that the source dimensions are sufficient to completely classify filtered Clifford supermodules, but there may be other information required as well.

2. N=5 EXAMPLES

Let $\{\gamma_1, \ldots, \gamma_5\}$ be the generators of the real Clifford algebra Cl(5). The two irreducible real representations of Cl(5) can be constructed as the quotients

$$\mathbb{S}_\pm = \mathrm{Cl}(5) \,/\, \langle \gamma_1\gamma_2\gamma_3\gamma_4 = \pm 1 \rangle.$$

As a basis for such a spin representation, we can take:

$$\begin{aligned} \text{even:}\quad & 1 \\ \text{odd:}\quad & \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5 \\ \text{even:}\quad & \gamma_1\gamma_2,\ \gamma_1\gamma_3,\ \gamma_1\gamma_4,\ \gamma_1\gamma_5,\ \gamma_2\gamma_5,\ \gamma_3\gamma_5,\ \gamma_4\gamma_5 \\ \text{odd:}\quad & \gamma_1\gamma_2\gamma_5,\ \gamma_1\gamma_3\gamma_5,\ \gamma_1\gamma_4\gamma_5 \end{aligned}$$

All other products of the Clifford generators γ_i can be identified with these, up to a sign. For instance $\gamma_2\gamma_4 \sim \gamma_1\gamma_3$ and $\gamma_2\gamma_3\gamma_4 \sim \gamma_1$. Note that there are eight even bosons and eight odd fermions, as expected. The Clifford algebra acts on the spin representation by left multiplication, and up to a sign, the action of the generator γ_i either adds or removes a γ_i factor from these basis elements. For instance

$$\gamma_2 \colon \gamma_1\gamma_3\gamma_5 \longmapsto \gamma_1\gamma_2\gamma_3\gamma_5 \sim \gamma_4\gamma_5.$$

Let us first consider a filtration with graded dimensions $1-8-7$. The F_0 component is one dimensional, and without loss of generality we can assume that F_0 is generated by the basis element 1. (If it were generated by another element ω, then we can simply right-multiply all the basis elements by ω.) The graded components of the filtration degrees then have bases:

$$\begin{aligned} \mathrm{Gr}_0 \colon\quad & \boxed{1} \\ \mathrm{Gr}_1 \colon\quad & \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \boxed{\gamma_1\gamma_2\gamma_5,\ \gamma_1\gamma_3\gamma_5,\ \gamma_1\gamma_4\gamma_5} \\ \mathrm{Gr}_2 \colon\quad & \gamma_1\gamma_2,\ \gamma_1\gamma_3,\ \gamma_1\gamma_4,\ \gamma_1\gamma_5,\ \gamma_2\gamma_5,\ \gamma_3\gamma_5,\ \gamma_4\gamma_5 \end{aligned}$$

Here I have boxed those basis elements which are not required by the filtration condition $\gamma_i F_p \subset F_{p+1}$, or in other words those which are non-trivial in the quotient $F_{p+1} \,/\, \mathbb{R}^5 F_p$. These boxed elements correspond to the sources in an Adinkra. We would say that this filtration has source dimensions $1-3-0$.

Next, let us consider filtrations with graded dimensions $2-8-6$. In this case, F_0 is two dimensional, and by the argument given above, we can take one of the basis elements of F_0

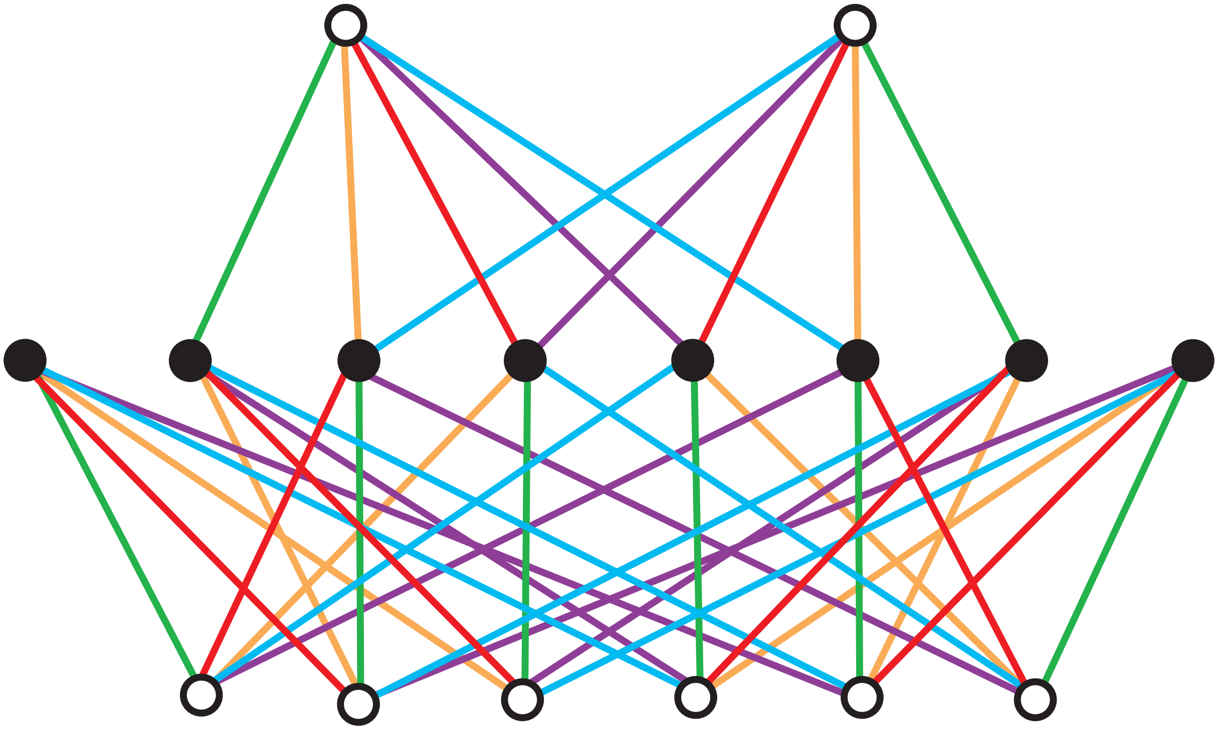
to be 1. The second basis element can be any other even element. It turns out there are two fundamentally distinct possibilities given in terms of our basis elements:

$$\begin{array}{ll}
\mathrm{Gr}_0: & \boxed{1,\ \gamma_1\gamma_2} \\
\mathrm{Gr}_1: & \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_1\gamma_2\gamma_5,\ \boxed{\gamma_1\gamma_3\gamma_5,\ \gamma_1\gamma_4\gamma_5} \\
\mathrm{Gr}_2: & \gamma_1\gamma_3,\ \gamma_1\gamma_4,\ \gamma_1\gamma_5,\ \gamma_2\gamma_5,\ \gamma_3\gamma_5,\ \gamma_4\gamma_5
\end{array}$$

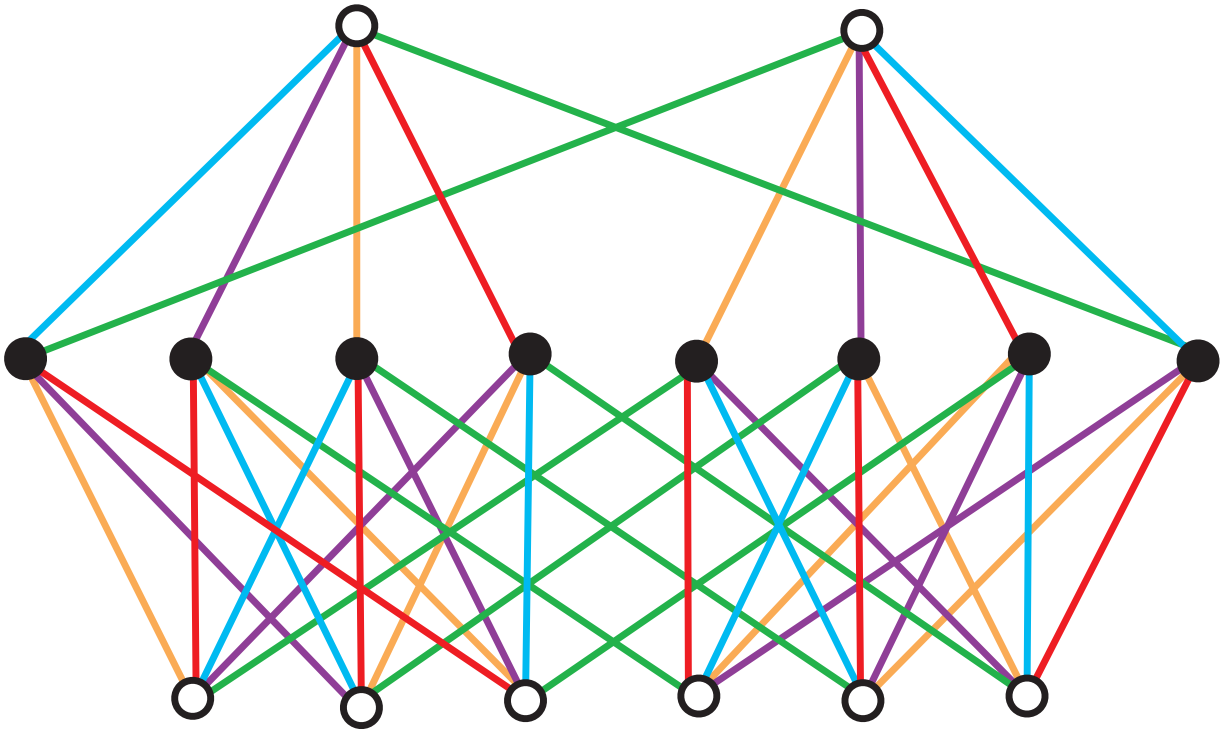
and

$$\begin{array}{ll}
\mathrm{Gr}_0: & \boxed{1,\ \gamma_1\gamma_5} \\
\mathrm{Gr}_1: & \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_1\gamma_2\gamma_5,\ \gamma_1\gamma_3\gamma_5,\ \gamma_1\gamma_4\gamma_5 \\
\mathrm{Gr}_2: & \gamma_1\gamma_3,\ \gamma_1\gamma_4,\ \gamma_1\gamma_5,\ \gamma_2\gamma_5,\ \gamma_3\gamma_5,\ \gamma_4\gamma_5
\end{array}$$

We can see that these two cases have different structures since there is no box in Gr_1 of the second one. In terms of Adinkras, the first case has two fermionic sources, while the second case has no fermionic sources. While both of these filtrations have graded dimensions $2-8-6$, the source dimensions are $2-2-0$ and $2-0-0$, respectively. Here are the two distinct Adinkras:



and



respectively.

We note that if we take our second basis element in F_0 to be $\gamma_1\gamma_3$ or $\gamma_1\gamma_4$, then we get the same picture as the first case simply by relabeling γ_2 as γ_3 or γ_4. This amounts to changing the colors of the edges in the Adinkra but otherwise leaving its structure unchanged. If we take our second basis element in F_0 to be $\gamma_i\gamma_5$ for $i = 2, 3, 4$, then we get the same picture as the second case, again by relabeling γ_1 as γ_i.

Why do we get two distinct filtrations and Adinkras? It is because constructing the irreducible real representations of Cl(5) involves choosing a distinguished degree 4 element $\gamma_1\gamma_2\gamma_3\gamma_4$, or equivalently by choosing a distinguished Clifford generator γ_5 dual to it. The first case involved $\gamma_i\gamma_j$ with $i, j \neq 5$, while the second case involved $\gamma_i\gamma_5$ with $i \neq 5$. By shuffling the γ-indices $1, 2, 3, 4$, we get the same pictures with different colors, but the γ-index 5 must always be treated differently. In both of the above Adinkras, the Clifford generator γ_5 is represented by the green edges.

But wait, what if we consider a basis for F_0 given by 1 and $\gamma_1(\gamma_2+\gamma_5)$ (possibly normalized by dividing it by $\sqrt{2}$)? There is no way we could obtain $\gamma_1(\gamma_2 + \gamma_5)$ by sequentially acting on 1 by Clifford generators γ_i. Also, we cannot relabel the Clifford generators by setting, say, $\gamma_2' = \gamma_2 + \gamma_5$, since the choice of representation fixes γ_5. So, we cannot write this F_0 in terms of a suitable basis, and thus any filtration with this F_0 is NOT Adinkrizable!

In fact, we can construct a whole family of non-Adinkrizable filtrations, with basis

$$F_0 : \quad 1,\ \gamma_1(\gamma_2 \sin\theta + \gamma_5 \cos\theta),$$

for $0 < \theta < \pi/2$. For $\theta = \pi/2$, we get our first case above, with $\gamma_1\gamma_2$, and for $\theta = 0$, we get our second case above, with $\gamma_1\gamma_5$. Using such choices of F_0, we can construct filtrations with graded dimensions $2-8-6$, and provided that $\sin\theta$ and $\cos\theta$ do not vanish, they all have source dimensions $2-0-0$ (you can check this for yourself). Thus, this whole family has the same basic type as the second case with $\gamma_1\gamma_5$ considered above. On the other hand, when $\theta=\pi/2$, the filtration jumps to the first case.

3. A smooth stratification indexed by Adinkras

The discussion of the examples in the previous section suggests the following:

Definition 3.1. Let V_1 and V_2 be two filtered $\mathrm{Cl}(N)$ supermodules. We say that V_1 and V_2 are *equivalent* if there exists vector space isomorphisms

$$f\colon V_1\to V_2, \qquad g:\mathbb{R}^N\to\mathbb{R}^N$$

such that

$$f(F_pV_1)=F_pV_2$$

(this is called a *proper* morphism of filtered spaces), and

$$g(\gamma)f(v)=f(\gamma v)$$

for $\gamma\in\mathbb{R}^N$ and $v\in V$.

This definition of equivalence allows us to relabel or redefine the γ_i, provided that we preserve the action on the representation. Note that without allowing the map g, we would be severely constrained by Schur's lemma if we were considering an irreducible representation of $\mathrm{Cl}(N)$. (Note that Schur's lemma says that the only automorphisms of an irreducible complex representation are constant multiples of the identity. For real representations, Schur's lemma is slightly more complicated, but it still restricts you to two dimensions worth of automorphisms.)

Definition 3.2. The *moduli space* of off-shell representations of $1D$ supersymmetry consists of filtered Clifford supermodules modulo equivalence.

This is very much an algebro-geometric definition. After all, a filtered Clifford supermodule is a flag

$$F_0V\subset F_1V\subset\cdots\subset V$$

satisfying the Clifford compatibility condition $\gamma_iF_pV\subset F_{p+1}V$. The space of flags with fixed dimensions on a given vector space V is called a *flag manifold*. Such manifolds are homogeneous spaces which can be analyzed via algebraic geometry. To construct our moduli space, we take a constrained subset of a flag manifold and then quotient by an equivalence relation, which has the flavor of moduli spaces in geometry.

Conjecture 3.3. *The moduli space of equivalence classes of off-shell representations of* $1D$ *supersymmetry forms a topological space. Every representation can be smoothly deformed among representations of the same "type" (as encoded by source dimensions) into one that is Adinkrizable. Furthermore, the set of all such representations for a given Adinkra is open, and the corresponding Adinkra may jump at the boundary of the set.*

This structure is called a *smooth stratification*. In other words, the moduli space is the disjoint union of open sets, called strata, indexed by Adinkras. Furthermore, these strata are glued along their boundaries to other strata (like in a CW complex, but where the strata are not necessarily discs). Preferably, the strata should be partially ordered, so that a stratum is glued only to lower strata.

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