The Formulas of Vector Calculus

John Cullinan

Analytic Geometry

A <u>vector</u> $\mathbf{v}$ is an n-tuple of real numbers: $\mathbf{v} = (v_1, \ldots, v_n)$. Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$, addition and multiplication with a scalar $t \in \mathbf{R}$ are defined by

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (v_1, \ldots, v_n) + (w_1, \ldots, w_n) = (v_1 + w_1, \ldots, v_n + w_n) \\ t \cdot \mathbf{v} &= t \cdot (v_1, \ldots, v_n) = (tv_1, \ldots, tv_n). \end{aligned}$$

Here is a brief list of definitions

(1) The <u>dot product</u>: $\mathbf{v} \cdot \mathbf{w} = (v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) = \sum_{i=1}^{n} v_i w_i$.

(2) The <u>length</u> of a vector: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$. Vectors of length 1 are called <u>unit vectors</u>.

(3) The <u>angle</u> θ between two vectors: $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$. For two vectors $\mathbf{v}$, $\mathbf{w}$ with $\|\mathbf{v}\|, \|\mathbf{w}\| \neq 0$, the enclosed angle θ is hence given by

$$\theta = \cos^{-1} \frac{\sum_{i=1}^{n} v_i w_i}{\left(\sum_{i=1}^{n} v_i^2\right)^{1/2} \left(\sum_{i=1}^{n} w_i^2\right)^{1/2}}$$

(4) Two vectors $\mathbf{v}, \mathbf{w}$ are called
 (a) <u>perpendicular</u> or <u>orthogonal</u> if $\mathbf{v} \cdot \mathbf{w} = 0$. We denote this by $\mathbf{v} \perp \mathbf{w}$.
 (b) <u>parallel</u> if $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\|$. We denote this by $\mathbf{v} \| \mathbf{w}$.

Note that two vectors $\mathbf{v}, \mathbf{w}$ are parallel if and only if there is a scalar $t \in \mathbf{R}$ such that $\mathbf{v} = t \cdot \mathbf{w}$.

The zero vector $\mathbf{0} = (0, \ldots, 0)$ is by definition parallel *and* perpendicular to every vector.

<u>Lines and Planes</u> Let $\mathbf{u}, \mathbf{v}$ be two vectors with $\mathbf{v} \neq \mathbf{0}$. Then

$$\mathbf{u} + t \cdot \mathbf{v} = (u_1 + tv_1, \ldots, u_n + tv_n); \ t \in \mathbf{R}$$

yields a line in $\mathbf{R}^n$. This line passes through the point $\mathbf{u}$ in the direction of $\mathbf{v}$.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ be vectors where $\mathbf{v}$ and $\mathbf{w}$ are *not* parallel. Then

$$\mathbf{u} + s \cdot \mathbf{v} + t \cdot \mathbf{w} = (u_1 + sv_1 + tw_1, \ldots, u_n + sv_n + tw_n)$$

defines a plane through the point $\mathbf{u}$, spanned by $\mathbf{v}$ and $\mathbf{w}$.

The notion of a perpendicular vector yields another form of representing a plane in $\mathbf{R}^3$. Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ be a fixed point in the plane and $\mathbf{n}$ a vector perpendicular to the plane. An arbitrary point $\mathbf{r} = (x, y, z)$ on the plane satisfies

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

Moreover, the equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ defines a plane in $\mathbf{R}^3$ for *any* fixed $\mathbf{n}, \mathbf{v} \in \mathbf{R}^3$, provided $\mathbf{n} \neq \mathbf{0}$.

In $\mathbf{R}^3$, the representation of a line $\ell(t) = (x, y, z) = \mathbf{u} + t \cdot \mathbf{v}$ can be solved for t in each coordinate if none of the values v_1, v_2, v_3 are zero:

$$t = \frac{x - u_1}{v_1} \qquad t = \frac{y - u_2}{v_2} \qquad t = \frac{z - u_3}{v_3}$$

This yields another representation of a line in $\mathbf{R}^3$, which is called a *symmetric equation*:

$$\frac{x - u_1}{v_1} = \frac{y - u_2}{v_2} = \frac{z - u_3}{v_3}.$$

The parametric equation representing a plane can be transformed similarly giving the standard equation of a plane:

$$ax + by + cz = d,$$

where the normal vector $\mathbf{n}$ is (a, b, c).

Cross Product At a point p of any plane in $\mathbf{R}^3$ there is exactly one line perpendicular to the plane. If $\mathbf{v}$ and $\mathbf{w}$ span the plane, a vector spanning this perpendicular line is given by the cross product $\mathbf{v} \times \mathbf{w}$.

For two vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ the *cross product* $\mathbf{v} \times \mathbf{w}$ is defined by

$$\mathbf{v} \times \mathbf{w} = (v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

Alternatively, the cross product is given by the 3×3 determinant $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

- $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$
- $\mathbf{v}, \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ are oriented according to the right hand rule
- if θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$.

The length of $\mathbf{v} \times \mathbf{w}$ is equal to the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

The cross product is

(1) *linear*: $t \cdot (\mathbf{v} \times \mathbf{w}) = (t \cdot \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (t \cdot \mathbf{w})$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
(2) *anti-commutative:* $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

Another remarkable property of the cross product is the following:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

Geometrically, $\|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\|$ is the volume of the parallelepiped given by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

Spheres and Cylinders The vector notation allows for an elegant description of other geometric objects such as spheres and cylinders.

- For a scalar $r \geq 0$ and a vector $\mathbf{c} \in \mathbf{R}^3$, the equation $\|\mathbf{x} - \mathbf{c}\|^2 = r^2$ yields a *sphere* of radius r centered at $\mathbf{c}$.
- For a scalar $r \geq 0$ and vectors $\mathbf{c}, \mathbf{n} \in \mathbf{R}^3$ with $\|\mathbf{n}\| = 1$, the equation $\|\mathbf{n} \times (\mathbf{x} - \mathbf{c})\|^2 = r^2$ yields a *cylinder* of radius r centered around the line given by $\mathbf{c} + t \cdot \mathbf{n}$.

Parameterized Curves in $\mathbf{R}^3$

Velocity, Speed, and Acceleration Given three differentiable functions $x, y, z : [a, b] \to \mathbf{R}$, the vector

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

can be understood as describing a particle moving through space; the particle's position depends on the (time) parameter t. This perception gives rise to the following definitions. We define the

- *velocity* of $\mathbf{r}$ at time t to be $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$
- *speed* of $\mathbf{r}$ at time t to be $\|\mathbf{r}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$
- *acceleration* of $\mathbf{r}$ at time t to be $\mathbf{r}''(t) = (x''(t), y''(t), z''(t))$.

Particles moving at constant speed Suppose that $\|\mathbf{r}(t)\| = c$ for all t for curve $\mathbf{r}(t)$. Differentiating $\|\mathbf{r}(t)\| = c$, we observe that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t: $\|\mathbf{r}(t)\| = 1 \Rightarrow \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for all t.

Smooth Curves A curve $\mathbf{r} : [a, b] \longrightarrow \mathbf{R}^3$ is called smooth if its speed vanishes at most at the endpoints:

$$|\mathbf{r}'(t)| \neq 0 \text{ for all } t \in (a, b).$$

Arclength The arclength $\ell(a, b)$ of a curve $\mathbf{r} : [a, b] \to \mathbf{R}^3$ is given by

$$\ell(a, b) = \int_a^b \|\mathbf{r}'(t)\| \, \mathrm{d}t$$

A curve $\mathbf{r} : [a, b] \to \mathbf{R}$ is called parameterized with respect to arclength if $\|\mathbf{r}'(t)\| = 1$ for all $t \in [a, b]$.

Curvature Let $\mathbf{r} : [a, b] \to \mathbf{R}^3$ be parameterized with respect to arclength. Then $\kappa(t) = \|\mathbf{r}''(t)\|$ is called the curvature of $\mathbf{r}$ at t. Differentiating the equation $\|\mathbf{r}'(t)\| = 1$ shows that $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are orthogonal for all t. If $\kappa(t) \neq 0$, the vector

$$\mathbf{n}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|}$$

is a well-defined unit vector.

Normal and Binormal Vectors If $\kappa(t) \neq 0$, we can define the

- *Normal vector* of the curve at time t to be $\mathbf{n}(t)$
- *Binormal vector* of the curve at time t to be $\mathbf{b}(t) = \mathbf{r}'(t) \times \mathbf{n}(t)$

The binormal vector is obviously a unit vector, so we can apply the same reasoning as before to see that $\mathbf{b}(t)$ and $\mathbf{b}'(t)$ are orthogonal. On the other hand, differentiating $\mathbf{b}(t) = \mathbf{r}'(t) \times \mathbf{n}(t)$ we get:

$$\mathbf{b}'(t) = \mathbf{r}''(t) \times \mathbf{n}(t) + \mathbf{r}'(t) \times \mathbf{n}'(t) = \mathbf{r}'(t) \times \mathbf{n}'(t).$$

Hence $\mathbf{b}'(t)$ is parallel to $\mathbf{n}(t)$.

Torsion The equation $\mathbf{b}'(t) = \tau(t) \cdot \mathbf{n}(t)$ defines the torsion τ of the curve at time t.

Frenet Frame Whenever $\kappa(t) \neq 0$, the vectors $\mathbf{r}'(t), \mathbf{n}(t), \mathbf{b}(t)$ are mutually orthogonal unit vectors. They span the Frenet Frame.

Curvature of Curves NOT parmeterized with respect to Arclength Let $\mathbf{r} : [a, b] \to \mathbf{R}^3$ be *amy* smooth curve. Its curvature $\kappa(t)$ at time t is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Real-valued Functions

Domain and Range A function $f : \mathbf{R}^n \to \mathbf{R}$ assigns a unique real value $f(x_1, \ldots, x_n)$ to each point $(x_1, \ldots, x_n)$ of a set D in $\mathbf{R}^n$. The set D is called the domain of f. The set $R = \{f(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in D\}$ is called the range of f.

Limits A real number L is said to be the *limit* of f at $(a, b, \ldots)$ if *for all* sequences $(a_m, b_m, \ldots)$ with $\lim_{m\to\infty} a_m = a$, $\lim_{m\to\infty} b_m = b$, ..., the following holds:

$$\lim_{m\to\infty} f(a_m, b_m, \ldots) = L.$$

We denote this by

$$\lim_{(x_1,x_2,\ldots)\to(a,b,\ldots)} f(x_1, x_2, \ldots) = L.$$

Equivalently, if for every real number $\epsilon > 0$ there is another real number $\delta > 0$ such that

$$\|(x_1, x_2, \ldots) - (a, b, \ldots)\| < \delta \Rightarrow |f(x_1, x_2, \ldots) - L| < \epsilon.$$

Continuity A function $f : \mathbf{R}^n \to \mathbf{R}$ with domain D is said to be continuous at $(a, b, \ldots) \in D$ if

$$\lim_{(x_1,x_2,\ldots)\to(a,b,\ldots)} f(x_1, x_2, \ldots) = f(a, b, \ldots)$$

Partial Derivatives Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function with domain D and $(x_1, \ldots, x_n) \in D$. The partial derivative of f at $(x_1, \ldots, x_n)$ *with respect to* x_i is given by the limit

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

Differentiability A function $f : \mathbf{R}^n \to \mathbf{R}$ with domain D is called differentiable at $(x_1, \ldots, x_n) \in D$ if *all* partial derivatives exist *and are continuous* at $(x_1, \ldots, x_n)$.

Directional Derivaitve Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function with domain D and $\mathbf{r} \in D$. Suppose that $\mathbf{u}$ is a unit vector in $\mathbf{R}^n$. The directional derivative of f at $\mathbf{r}$ in the direction of $\mathbf{u}$ is given by

$$\left.\frac{d}{dt} f(\mathbf{r} + t\mathbf{u})\right|_{t=0}.$$

Gradient Vector Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function with domain D that is differentiable at $\mathbf{r} \in D$. The gradient vector ∇f of f at $\mathbf{r}$ is given by

$$\nabla f\big|_{\mathbf{r}} = \left(\left.\frac{\partial f}{\partial x_1}\right|_{\mathbf{r}}, \ldots, \left.\frac{\partial f}{\partial x_n}\right|_{\mathbf{r}} \right)$$

Directional Derivatives and the Gradient Vector Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function with domain D that is differentiable at $\mathbf{r} \in D$. Using the chain rule, the directional derivative is given by

$$\frac{d}{dt} f(\mathbf{r} + t\mathbf{u})\Big|_{t=0} = \nabla f\big|_{\mathbf{r}} \cdot \mathbf{u}$$

Higher Derivatives and Clairaut's Theorem Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function with domain D and suppose the partial derivatives of f are themselves differentiable. Then differentiating $\frac{\partial f}{\partial x_i}$ with respect to x_j is the same as differentiating $\frac{\partial f}{\partial x_j}$ with respect to x_i:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Global and Local Extrema A function $f : \mathbf{R}^n \to \mathbf{R}$ with domain D has

- a *global maximum* at $\mathbf{r} \in D$ if $f(\mathbf{x}) \leq f(\mathbf{r})$ for all $\mathbf{x} \in D$
- a *global minimum* at $\mathbf{r} \in D$ if $f(\mathbf{x}) \geq f(\mathbf{r})$ for all $\mathbf{x} \in D$
- a *local maximum* at $\mathbf{r} \in D$ if there is a disc R centered at $\mathbf{r}$ such that $f(\mathbf{x}) \leq f(\mathbf{r})$ for all $\mathbf{x} \in R$
- a *local minimum* at $\mathbf{r} \in D$ if there is a disc R centered at $\mathbf{r}$ such that $f(\mathbf{x}) \geq f(\mathbf{r})$ for all $\mathbf{x} \in R$

Critical Points Let $f : D \subset \mathbf{R}^n \to \mathbf{R}$ be differentiable. We call a point $\mathbf{r} \in D$ a *critical point* if $\nabla f\big|_{\mathbf{r}} = \mathbf{0}$. If f has an extremum at $\mathbf{r}$, then $\mathbf{r}$ is critical, but the converse is not necessarily true.

Functions of Two Variables – Second Derivative Test Let $f : D \subset \mathbf{R}^2 \to \mathbf{R}$ and its derivatives be differentiable and let $(x_0, y_0) \in D$ be a critical point of f. Let

$$D(x_0, y_0) = \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right) \Bigg|_{(x_0, y_0)}$$

- if $D(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} < 0$, then f has a maximum at (x_0, y_0).
- if $D(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} > 0$, then f has a minimum at (x_0, y_0).
- if $D(x_0, y_0) < 0$, then f has a saddle at (x_0, y_0).
- if $D(x_0, y_0) = 0$, then the second derivative test gives no information about the nature of the critical point.

The Double Integral Let $R = [a, b] \times [c, d]$ and let $f : R \to \mathbf{R}$ be continuous. The *double integral* of f over R is defined to be

$$\iint_R f(x, y)\ \mathrm{d}A = \lim_{|P| \to 0} \sum_{i,j} (x_i - x_{i-1})(y_j - y_{j-1}) f(x_i^*, y_j^*)$$

where $P = P_{[a,b]} \times P_{[c,d]}$ is a partition of R, $|P| = |P_{[a,b]}| \cdot |P_{[c,d]}|$ is its norm, and $(x_i, y_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Fubini's Theorem Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous and $R = [a, b] \times [c, d]$. The double integral is given by

$$\iint_R f(x, y)\ \mathrm{d}A = \int_a^b \int_c^d f(x, y)\ \mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\ \mathrm{d}x\mathrm{d}y$$

Level Surfaces Let $f(x, y, z) : \mathbf{R}^3 \to \mathbf{R}$ be a function of three variables. The level set $\{(x, y, z) \mid f(x, y, z) = c\}$ for a constant c generally yields a surface in $\mathbf{R}^3$.

Tangent Planes of Level Surfaces Consider the level set $\{(x, y, z) \mid f(x, y, z) = c\}$ of a function $f : \mathbf{R}^3 \to \mathbf{R}$.

(1) The gradient vector $\nabla f(x_0, y_0, z_0)$ at a point (x_0, y_0, z_0) is perpendicular to the plane tangent to the level surface at (x_0, y_0, z_0).

(2) The tangent plane is given by the equation

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Using Lagrange Multipliers to Find Extrema with Constraints Let $f : \mathbf{R}^3 \to \mathbf{R}$ be differentiable. To find the maximum and minimum value of f subject to the constraint

$$g(x, y, z) = c,$$

the gradients ∇f and ∇g must be parallel. An algorithm to find the maximum and minimum values is hence given by:

(1) Find all points (x, y, z) such that $\nabla f = \lambda \nabla g$, for some $\lambda \in \mathbf{R}$, and $g(x, y, z) = c$.
(2) Evaluate f at these points. The largest (smallest) value is the maximum (minimum) of f subject to the constraint $g(x, y, z) = c$.

The Triple Integral Let $g : \mathbf{R}^3 \to \mathbf{R}$ be continuous and $R = [a, b] \times [c, d] \times [e, f]$. The *triple integral* of g over R is defined to be

$$\iiint_R g(x, y, z) \ \mathrm{d}V = \lim_{|P| \to 0} \sum_{i,j,k} (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) g(x_i^*, y_j^*, z_k^*)$$

Chain Rule Let $\mathbf{r}(t)$ be a smooth curve in $\mathbf{R}^n$ and let $f : \mathbf{R}^n \to \mathbf{R}$ be a function of several variables. Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \nabla f \cdot \mathbf{r}'(t).$$

More generally, suppose each of the variables x_i is a function of the variables $t_1, \ldots, t_m$, then

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

Change of Variables Formula in Two Dimensions Let $(x(u, v), y(u, v))$ be a parameter transformation with Jacobian matrix

$$J = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

which maps $S \subset \mathbf{R}^2$ into $R \subset \mathbf{R}^2$. Then

$$\iint_R f(x, y) \ \mathrm{d}x\mathrm{d}y = \iint_S f(u, v) |\det J| \ \mathrm{d}u\mathrm{d}v$$

Polar Coordinates The parameter transformation $(x(r, \theta), y(r, \theta)) = (r \cos \theta, r \sin \theta)$ expresses (x, y) in *polar coordinates*. Then

$$\iint_R f(x, y) \ \mathrm{d}x\mathrm{d}y = \iint_S f(r, \theta) \ r\mathrm{d}r\mathrm{d}\theta$$

Surface Area Let $f : \mathbf{R}^2 \to \mathbf{R}$ be differentiable. The *surface area* of f over D is given by

$$\iint_R \sqrt{1 + \left[\frac{\partial f}{\partial x}\right]^2 + \left[\frac{\partial f}{\partial y}\right]^2} \ \mathrm{d}x\mathrm{d}y$$

Path Integrals Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be differentiable and let $\mathbf{r}(t)$ parameterize a smooth curve C in $\mathbf{R}^2$. The *path integral* of f along C is given by

$$\int_C f(\mathbf{r}(t)) \cdot \|\mathbf{r}'(t)\| \ \mathrm{d}t$$

Vector Fields

A <u>vector field</u> $\mathbf{F}$ assigns to each point in a domain $R \subset \mathbf{R}^n$ a vector in $\mathbf{R}^n$.

<u>Line Integrals</u> Let $\mathbf{F}$ be a vector field on a domain $R \subset \mathbf{R}^n$ and let C be a smooth curve in R parameterized by $\mathbf{r}(t)$. The *line integral* of $\mathbf{F}$ along C is given by

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ \mathrm{d}t$$

<u>Conservative Vector Fields and the Potential Function</u> If $\mathbf{F} = \nabla f$ for some function $f : \mathbf{R}^2 \to \mathbf{R}$, then $\mathbf{F}$ is called a *conservative vector field* with *potential function* f.

<u>Line Integrals of Conservative Vector Fields</u> Let $\mathbf{F} = \nabla f$ be conservative on a domain $R \subset \mathbf{R}^n$. For any continuous curve C in R from $\mathbf{u}$ to $\mathbf{b}$ which is parameterized by $\mathbf{r}(t)$, we have

$$\int_C \mathbf{F} \cdot \mathbf{r}'(t)\ \mathrm{d}t = f(\mathbf{v}) - f(\mathbf{u}).$$

In particular

(1) If C is closed then $\int_C \mathbf{F} \cdot \mathbf{r}'\ \mathrm{d}t = 0$.

(2) if $\widetilde{C}$ is another continuous curve in R from $\mathbf{u}$ to $\mathbf{v}$ parameterized by $\mathbf{s}(t)$, then $\int_{\widetilde{C}} \mathbf{F} \cdot \mathbf{s}'\ \mathrm{d}t = \int_C \mathbf{F} \cdot \mathbf{r}'\ \mathrm{d}t$. The integral is said to be *path-independent.*

<u>Line Integrals of Vector Fields in $\mathbf{R}^2$</u> Let $\mathbf{F} = (P, Q)$ be a vector field on a domain $R \subset \mathbf{R}^2$ and let C be a smooth curve in R given by $(x(t), y(t))$. Then

$$\int_a^b \mathbf{F}(P(x(t), y(t)), Q(x(t), y(t))) \cdot (x'(t), y'(t))\ \mathrm{d}t = \int_C P(x, y)\ \mathrm{d}x + \int_C Q(x, y)\ \mathrm{d}y$$

<u>Conservative Vector Fields in $\mathbf{R}^2$</u> If $\mathbf{F} = (P, Q) = \nabla f$, then by Clairaut's theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

<u>Green's Theorem</u> Let R be a simply connected region with positively-oriented boundary ∂R. Then

$$\int_{\partial R} P\ \mathrm{d}x + Q\ \mathrm{d}y = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ \mathrm{d}x\mathrm{d}y$$

<u>Stokes' Theorem</u> If S is a surface in $\mathbf{R}^3$ with boundary ∂S parameterized by $\mathbf{r}$ and $\mathbf{F}$ is a vector field in $\mathbf{R}^3$, then

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

<u>Divergence (Gauss') Theorem</u> If V is a compact volume in $\mathbf{R}^3$ with boundary $\partial V = S$ and $\mathbf{F}$ is a vector field in $\mathbf{R}^3$, then

$$\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$