THE DISCRIMINANT OF A COMPOSITION

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This lemma may be useful in the following contexts: a) to compute the discriminant of an iterated rational function, or b) to extract the discriminant of f or g from that of $f \circ g$. Let k be a field and $f, g \in k[x]$. Suppose $\deg(f) = \epsilon$ and $\deg(g) = \delta$. Suppose further that neither f nor g has zero discriminant and that $\operatorname{Res}(f, g) \neq 0$, *i.e.* f and g have no common roots.

Lemma. *With all notation as above, the discriminant of the composition $f \circ g$ is given by the formula*

$$\operatorname{disc} f \circ g = (-1)^{\epsilon\delta(3\epsilon\delta-2\delta-1)/2} \ell(f)^{\delta-1} \ell(g)^{\epsilon(\epsilon\delta-\delta-1)} (\operatorname{disc} f)^{\delta} \operatorname{Res}(f \circ g, g'),$$

where ℓ denotes the leading term of the polynomial.

Proof. The definition of discriminant in terms of the resultant is gives us

$$\operatorname{disc} f \circ g = (-1)^{\binom{\epsilon\delta}{2}} \ell(f \circ g)^{-1} \operatorname{Res}(f \circ g, (f' \circ g)g').$$

Applying the identity $\operatorname{Res}(P, QR) = \operatorname{Res}(P, Q) \operatorname{Res}(P, R)$, we focus on the resultant $\operatorname{Res}(f \circ g, (f' \circ g)g')$:

$$\begin{aligned}
\operatorname{Res}(f \circ g, (f' \circ g)g') &= (-1)^{\epsilon\delta((\epsilon-1)\delta)} \ell(f' \circ g)^{\epsilon\delta} \prod_{\{\theta \,:\, f'(g(\theta))=0\}} f(g(\theta)) \\
&= (-1)^{\epsilon\delta((\epsilon-1)\delta)} \ell(f' \circ g)^{\epsilon\delta} \left[\prod_{\{\rho \,:\, f'(\rho)=0\}} f(\rho) \right]^{\delta} \\
&= (-1)^{\epsilon\delta((\epsilon-1)\delta)} \ell(f' \circ g)^{\epsilon\delta} \left[(\epsilon \ell(f)^{-\epsilon} \operatorname{Res}(f, f') \right]^{\delta} \\
&= (-1)^{\epsilon\delta((\epsilon-1)\delta)} \ell(f' \circ g)^{\epsilon\delta} (\epsilon \ell(f)^{-\epsilon\delta} \ell(f)^{\delta} (\operatorname{disc} f)^{\delta}.
\end{aligned}$$

Putting this resultant back into the discriminant formula above and working out the leading terms yields the desired formula. ☐

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