Writing a SENIOR PROJECT in MATHEMATICS

The Mathematics Program
Bard College

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The senior project at Bard is the culminating experience of your four years at the college. It will most likely be the longest and most complex piece of academic work you have done. It will also be one of the most memorable parts of your Bard education when you look back at your education many years from now. In the middle of it, however, a project can also be a source of frustration and anxiety, especially if you are unsure what you are supposed to be doing. This pamphlet is designed to let you know what doing a senior project in mathematics entails; it also gives some very specific information about formatting and the like. Please ask your adviser any questions you may have about senior projects.

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What is a Mathematics Senior Project

A senior project in mathematics is a year-long investigation of a particular topic or problem in mathematics. Such a topic may be in pure or applied mathematics. It might involve areas that relate to mathematics (e.g. physics or economics), and it might make use of computers, though in either case the project must be solidly mathematical. A project will involve at learning some new mathematics, working on a particular problem chosen by you and your adviser, and writing an original exposition of what you have learned. A project often involves attempting to prove some previously unknown result, though unlike a doctorate new results are not needed for a successful project. To get a feel for what a senior project in mathematics might be you might ask your adviser to look at some recent mathematics senior projects. Senior projects in mathematics are evaluated according to three criteria: mathematical content, exposition and effort.

1.1 Project Adviser

It is up to you to choose a project adviser, as long as he or she agrees to work with you. Your project adviser need not be your academic adviser. You should choose your project adviser before the end of your junior year, both because you need your adviser to sign your registration card when you register for senior project, and so that you and your adviser can discuss what you should be looking at over the summer prior to your senior year. During your senior year your adviser will set up a regular schedule of meetings with you.

1.2 Project Board

Your project board will consist of your adviser and two other faculty members to be assigned by the NS & M Division. If there are any faculty members you would particularly like on your board you may ask your adviser to request that the division assign these people, though it is not guaranteed that the division will accept your proposal.

1.3 Choosing a Project Topic

The mathematics department encourages projects in any branch of mathematics. The best project topic is one that can sustain your interest for a whole year. Although your adviser will be glad to suggest topics for you, it is best if you have as much input as possible into the choice of topic. If you have a specific topic you would like to pursue, propose it to your adviser, who will help evaluate its feasibility. If you do not have a specific topic, you might suggest to your adviser some areas of mathematics that particularly interest you. In your advanced mathematics courses, were there any issues mentioned that caught your attention? A good place to find mathematical topics is to browse through the mathematics collection in the library (which have Library of Congress letters QA). There are also a number of journals in the library that are aimed at undergraduates, including The American Mathematical Monthly, Mathematics Magazine and The Mathematical Intelligencer.

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Events Along the Way

There are a number of stages to be passed through during the completion of a senior project.

2.1 Prospectus Talk

During the first semester of the senior project you will have to give a 5 minute presentation to the students and faculty of the Natural Science and Mathematics Division about what you intend to do for your project. It is understood that anything you say at this point is tentative. The talk will be scheduled by the NS & M Division.

2.2 Midway Review

Halfway through the semester (usually right after winter break), your project board will meet with you to discuss your work on the project so far, and to make suggestions for your work for the rest of the academic year. If you have a draft of part of your project already written, you should circulate that draft to the board members prior to the midway review. To schedule your midway review, contact your adviser.

2.3 Handing in Drafts

You should set up with your adviser a schedule for handing in rough drafts. These drafts need to be handed in sufficiently ahead of the project due date so that your adviser will have time to read your drafts and make comments.

2.4 Handing in the Final Draft

On the date senior projects are due you must have three copies of your project ready; the Dean's Office will give all senior instructions on how to hand in your project. Make sure to allow enough time for printing and copying. The copies of your project that are due on the senior project due date need not be bound (though putting them in a loose-leaf binder is very convenient for the board members). After the board you might have to incorporate some corrections into your project. You should give a revised final draft to your adviser and each board member, and one copy needs to be turned into the library prior to graduation.

2.5 Final Project Board Meeting

After your project has been handed in on the date senior projects are due, your project board will meet with you for about an hour and a half to discuss you project. The grade for the project is determined by the NS & M Division after hearing a recommendation from the board. To schedule your final project board meeting, contact your adviser.

2.6 Poster Session

After the senior project due date, all seniors in the NS & M Division participate in the divisional poster session, in which each student presents a poster explaining the results obtained during the course of the senior project. Information about the poster session will be circulated to all seniors in the division.

3
Typing Mathematics

The problem with typing mathematics is the need for unusual symbols (such as Greek letters) and complicated formulas (such as matrices and subscripts of subscripts). Different branches of mathematics require differing levels of sophistication when it comes to symbols and formulas (compare a book on classical Greek geometry with a text on complex analysis), and the same holds for senior projects (there is, of course, no correlation between the difficulty of the mathematics and the nature of the symbols used).

There are two suggested methods for typing mathematics senior projects: either using a standard word processor (such as Microsoft WORD) or using TEX. The former is easier, but limited in terms of what symbols and formulas can be used; the latter is the state-of-the-art mathematical typesetting system, but it is a bit harder to use than a word processor.

3.1 Using a Word Processor

Some standard word processors have basic capabilities for mathematical formulas and symbols, and it might be that your project can be typed on such a word processor. In Microsoft WORD, for example, there is a built-in equation editor (called Microsoft Equation), that can be accessed via the Insert/Object menu. Microsoft Equation has Greek letters, a number of standard mathematical symbols (such as $\subset$ and $\in$), and basic constructs such as fractions, roots, sums, integrals and matrices. Whether a word processor will suffice for your senior project depends upon the nature of the symbols and formulas you need to use.

3.2 Using TEX

TEXis the state-of-the-art mathematical type-setting system, used by many mathematics journals and book publishers; it is also used by the Bard mathematics and computer science faculty (and some other faculty members as well). This document, for example, is written in TEX. Because TEX is not a what-you-see-is-what-you-get program, and is instead more like a programming language, it takes a little bit of effort to learn to use it. Whether it is worth your while to learn TEX depends upon a number of factors. If you are planning on going to mathematics or computer science graduate school, then learning TEX is definitely worthwhile. If your senior project topic includes a lot of fancy symbols or formulas that would be difficult to type in a standard word processor, then TEX would probably be worthwhile. If you want your senior project to be really professional-looking, then TEX is best (however, keep in mind that senior projects are not judged by appearance, and please do not go to the effort of learning TEX just to try to impress anyone).

If you want to find out more about TEX, talk to any mathematics or computer science faculty member. At Bard we recommend using the LATEX dialect of TEX. A LATEX style file (called bardproj.sty) that automatically formats Bard senior projects, as well as a manual for this style file, are available for downloading at http://math.bard.edu/bloch/bardtex.htm. Bard has a site license for the program PCTEX, which is a fairly user friendly implementation of TEX for Windows; TEX is also available on our Linux machines.

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Formatting Mathematics

In addition to the standard senior project formatting, as described in guidelines from the Registrar's office, there are a number of formatting guidelines specific to mathematics. The writing in a senior project is of two types: formal and informal. Formal writing is for definitions, statements of theorems, proofs, examples and the like; informal writing is for motivation, intuitive explanations, description of the literature, etc. A good project will have both kinds of writing—the formal writing is to make sure that mathematical rigor is being followed, and the informal writing to explain why the topic is interesting and the project proceeds as it does. In particular, interspersing formal and informal writing makes for a readable project. The important point to keep in mind is not to confuse the two types of writing, or else each will fail to do what it is supposed to do.

To keep the formal and the informal writing separate a very specific format for writing mathematics senior projects is used. This format is consistent with current mathematical standards. A sample text written in this format has been appended to this handout. The basic idea of the format to be used is that all formal writing is kept separate from the rest of the writing. There are a number of basic categories of formal formatting.

4.1 Definitions

Each definition begins with a declaration (usually in bold), and ends with an end-of-category symbol (usually a small square, but other things are possible). Each new term being defined should be highlighted in bold; it is reasonable to have more than one term being defined inside a single definition. A definition can be more than one paragraph. An example of a formal definition is as follows.

Definition 4.1.1. A **smooth curve** in $\mathbb{R}^3$ is a smooth function $c\colon (a,b) \to \mathbb{R}^3$, where (a,b) is an interval in $\mathbb{R}$. For each $t \in (a,b)$ the **velocity vector** of the curve at t is the vector $c'(t)$, and the **speed** at t is the real number $\|c'(t)\|$. A curve is **unit speed** if $\|c'(t)\| = 1$ for all $t \in (a,b)$. △

4.2 Examples

The format is the same as for definitions. Formal examples, it should be noted, require just as much rigor as in a proof.

4.3 Theorems, Lemmas, Propositions, Corollaries, Etc.

Theorems, lemmas, propositions and corollaries are all formal statements of results that need to be proved. Deciding whether a given result is a theorem or a proposition or a lemma is often a matter of judgement, but the general scheme of things is as follows: theorems are the most important results; lemmas are results that are of little interest in their own right, but that are used to prove theorems; propositions are somewhere between theorems and lemmas; corollaries are results that can be deduced easily from some theorem or proposition.

Each theorem, etc. begins with a declaration (usually in bold); there is no end-of-category symbol, since in most cases such a statement is immediately followed by a proof. In books and journals it is common to highlight the entire statement of a theorem, etc. by italicizing its text (or underlining it), though you need not do so (a fancy typesetting system such as TEX does the italicizing automatically). Some noteworthy theorems have names, which are usually inserted after the number of the theorem. An example of a statement of a theorem is as follows.

Theorem 4.3.1 (Fundamental Theorem of Curves). *Let $\bar{\kappa}, \bar{\tau}\colon (a,b) \to \mathbb{R}$ be smooth functions with $\bar{\kappa}(t) > 0$ for all $t \in (a,b)$. Then there is a strongly regular unit speed curve $c\colon (a,b) \to \mathbb{R}^3$ whose curvature and torsion functions are $\bar{\kappa}$ and $\bar{\tau}$ respectively. If $c_1, c_2\colon (a,b) \to \mathbb{R}^3$ are two such curves, then c_2 can be obtained from c_1 by a rotation and translation of $\mathbb{R}^3$.*

4.4 Proofs

The format is similar to the format for definitions. Here is a simple theorem and proof.

Theorem 4.4.1. *Let a, b and c be integers. If $a|b$ and $b|c$, then $a|c$.*

Proof. Suppose that $a|b$ and $b|c$. Hence there are integers q and r such that $aq = b$ and $br = c$. Define the integer k by $k = qr$. Then $ak = a(qr) = (aq)r = br = c$. Since $ak = c$, we know that $a|c$. □

4.5 Formatting Rules

A number of formatting rules, designed to make the project easier to read, should be adhered to.

1. Each formal category (definitions, examples, etc.) should be preceded and followed by a blank line.

2. Definitions, examples and statements of results (but not proofs) should be numbered consecutively in each section. For example, in Section 3 of Chapter 2 one might have Definition 2.3.1, then Example 2.3.2, etc.

3. In mathematical writing the use of footnotes is discouraged. References should be embedded in the text, and can be referred to either by number (e.g. "see the proof of the above theorem in [12]") or by a brief abbreviation (e.g. "see [Smith90]").

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Figures

Figures, graphs, and the like are very helpful in making a mathematical text readable, and as such their use is encouraged. There are a number of ways of making figures and inserting them into the text, and a number of guidelines for using figures in mathematical texts.

5.1 Creating and Inserting Figures

The simplest way to create figures is to draw them by hand. That is really OK, even in today's high-tech world. To insert such figures, simply leave sufficient space in the text, and then paste the figures into the final printed draft of the senior project.

There are a number of programs that can be used to create figures on the computer. If you are using things such as graphs of functions, then you can create such figures in mathematical programs such as Mathematica (which Bard has available). Geometric figures are often best created in graphics programs. One very good program for creating mathematical illustrations is Adobe Illustrator, which is available on some campus machines (Adobe Photoshop, which is widely used in the arts at Bard, is not very useful for mathematics). Other programs can be used as well (though very simple programs such as Microsoft Paint are not of much use).

If you use a computer program to create figures, save them in a standard format, which makes them easier to insert into your senior project. If you are using a word processor, then JPEG is a good format to use (BMP is not as good—files are usually very large). If you are using TEX, then the best format in which to save figures is EPS (Encapsulated Post Script). For example, it is possible to save Adobe Illustrator files in EPS format; graphics in Mathematica can also be saved as EPS files. It is also possible to insert JPEG and other types of files into TEX, though doing so tends to depend upon the choice of implementation of TEX (whereas EPS appears to be quite univerally useable for TEX). For more details on inserting figures into a TEX file, see the manual for the bardproj.sty style file, which is available for downloading at http://math.bard.edu/bloch/bardtex.htm, or ask faculty members in mathematics and computer science.

5.2 Pointers for Using Figures in Mathematical Texts

A number of rules should be followed when using figures in a mathematical text. Keep in mind that the way figures work in other types of texts will not necessarily hold for mathematical text. The main thing to keep in mind is that in a mathematical text, the figures are not part of the written text, but are additional elements.

1. Every figure needs a label (e.g. "Figure 5.2.1," which would be the first figure in Section 5.2). The figure label should be directly below the figure. See the example below. (In TEX the figure numbers are done automatically; in a word processor you need to label the figures yourself.)

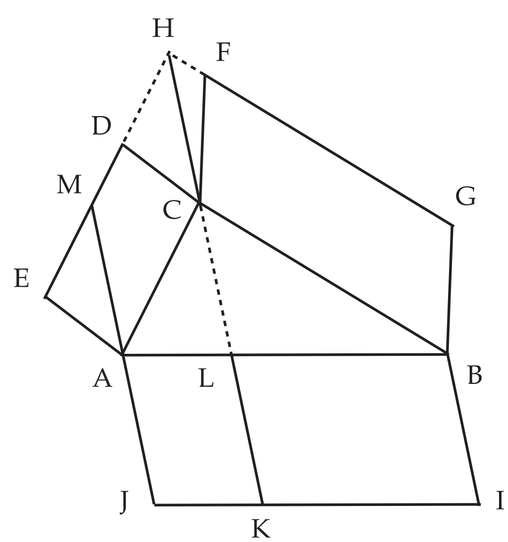

Figure 5.2.1

2. Every figure should be referred to in the written text (e.g. "see Figure 5.2.1"). The reader of a mathematical text will look at a figure when it is referred to; if you insert a figure that is not referred to in the text, then you cannot be sure that the reader will pay attention to it.

3. Every figure should be inserted immediately AFTER it is first referred to in the written text. If a figure is inserted prior to when it is first referred to, the reader will have no idea what to make of it.

4. Do not use descriptive captions—the description of the figure should be in the text. Indeed, it is best not to use captions at all, only figure numbers.

6
Writing Style

As in any other field, careful writing in mathematics is of great importance for both the writer—by promoting clear thinking—and the reader. Though mathematical thinking starts in one's head, any experienced mathematician knows that until an idea has been written up carefully one can never be sure of its correctness; putting an idea into written form forces the writer to pay attention to all the details of an argument. Often an idea that seemed to make sense in one's head is found to be lacking when put into written form. Since a project is meant to be read, it goes without saying that a project must be written in a clear style that the reader can follow.

A senior project is the first piece of lengthy mathematical writing that you will most likely have undertaken, and listed below are some standard principles of mathematical writing that are widely accepted in the mathematical community. Though some of these comments may seem like little more than cumbersome stylistic preferences, they actually make the writing more understandable to the reader. Please ask your adviser any questions you have about writing mathematics. The best way to learn to write mathematics is to hand in drafts to your adviser early enough so that he or she can make comments.

6.1 Prove What is Appropriate for a Senior Level Mathematics Course

A good proof should have just the right amount of detail—neither too little nor too much. The question of what needs to be included in a proof and what can be taken as known by the reader is often a matter of judgement, but a good guideline is to assume that the reader is at the exact same level of knowledge as a senior mathematics major at Bard—who is not writing a senior project on your topic. It is safe to assume that the reader knows the quadratic formula, for example, and no explanation of its use is needed. On the other hand, one should not assume that the reader knows anything beyond what has been covered in your mathematics courses. When in doubt—prove.

6.2 Formulating and Writing Proofs are Distinct Actitivities

There are two steps to the process of producing a rigorous proof: formulating the proof and writing it. The distinction between the activity of formulating a proof and writing it up cannot be overemphasized. Though in some very simple and straightforward proofs one might formulate as one writes, in most cases one first formulates the proof (at least in outline form) prior to writing. The write-up often does not resemble the original formulation. A proof is a step-by-step chain of reasoning from the given hypotheses to the desired conclusion. When formulating the proof one might start from the conclusion and work backwards, or one might start at the beginning and work forwards for a while, then work backwards from the conclusion, and hopefully things will meet in the middle. In writing a proof, by contrast, one should almost always start at the beginning and work straight through till the conclusion is reached. Not all arguments are reversible, and an argument that worked backwards during the formulation might not always work when written forwards. Intuitive thinking that may have been useful in formulating the proof should be replaced by logical deduction in the written proof. Think of the reader when you write a proof.

6.3 Prove Precise Statements

A proof cannot possibly be rigorous if the statement being proved is not precisely formulated. Mathematics is often read by skipping back and forth, and so it is important that the statements of theorems, lemmas, propositions and the like contain all their hypotheses, rather than having the hypotheses in some earlier paragraphs. Better a bit of redundancy than a confused reader.

6.4 Be Careful with Saying Things are "Obvious"

It is very tempting to skip over some details by saying that they are "obvious" or are "similar to what has already been shown." Such statements are legitimate if true, but are often used as a cover for uncertainty or laziness. "Obvious" is in the eye of the beholder; what may seem obvious to the writer after spending hours (or days) on a problem might not be so obvious to the reader, and since a proof should aim to convince the reader, that is by whom obviousness needs to be judged. That something is obvious should mean that another senior mathematics major could figure it out in very little time and with very little effort; if it does not conform to this criterion it is not "obvious." If something is truly obvious then there is probably no need to remind the reader of this fact. (By contrast, something is "trivial" if once thought of the proof of this thing is a simple logical deduction, though it might take quite a while to think of the proof; something can be trivial but not at all obvious.) It is much better to err on the side of too much detail than not enough.

6.5 Use Full Sentences and Correct Grammar

The use of correct grammar (for example, complete sentences and correct punctuation) is crucial if the reader is to follow what is written. The rules of grammar may at times seem arbitrary, but writing without grammar is like driving without obeying traffic lights—it may be fun but should never be undertaken if others are in the vicinity; if writing is meant to be read by others then communication, not thrills, should be the guiding principle. Mathematical symbols are shorthand for expressions that could just as well be written out in words, and thus symbols are subject to the rules of grammar just as words are. Mathematical symbols floating freely on a page are neither understandable nor acceptable. All symbols should be parts of sentences and paragraphs. See the attached sample mathematical writing for an illustration. Use words such as "therefore," "hence," "it follows that," and the like to link things up, as well as to guide the logical flow of an argument. The "two-column" approach to proofs often used in high school geometry class should be discarded (as should a number of other experiences one has in high school).

6.6 Define All Symbols and Terms You Make Up

Mathematics abounds in terminology and symbols, and one is often tempted to make up words and symbols of one's own, and to use all sorts of exotic alphabets. For the sake of readability, however, simplicity is the key. Do not use more symbols than absolutely necessary, and avoid exotic letters and complications (such as multiple subscripts) wherever possible. Try to stick to standard notation. If you must make up some notation make sure you define it explicitly. As always, one cannot assume that the reader is a mind-reader.

6.7 Break up a Long Proof into Steps

If a proof is long and difficult to follow, it is often wise to break it up into steps, or to isolate preliminary parts of the proof as lemmas (which are simply smaller theorems used to prove bigger theorems, but which are usually of little interest on their own). If you use lemmas be sure to state them precisely. Prior to going into the details of a long proof it is often useful to give a sentence or two outlining the strategy of the proof.

6.8 Intuitive Aids Help

Intuitive aids such as drawings, graphs, venn diagrams and the like are extremely helpful. However, intuitive aids should never be confused with a rigorous proof.

7
Sample Mathematical Writing

(excerpted from a discussion of continuity)

Intuitively, continuity is about a function having a graph that has no gaps or jumps. Examples show that this notion can be detected by looking at the openness or non-openness of inverse images of open interval; we take this observation as the basis for the following definition. We have not “proved” that the following definition corresponds exactly to our intuition about continuity, since one cannot prove intuitive things rigorously. The best one can hope for is that all desired intuitively reasonable properties hold, and all examples work out as expected; such is the case for the following definition.

Definition 7.1.1. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be sets, and let $f\colon A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. △

Observe that in the above definition it is only required that if a set U is open then $f^{-1}(U)$ is open; it is not required that whenever $f^{-1}(U)$ is open, then U is open.

Example 7.1.2. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be sets. The projection maps $\pi_1\colon A \times B \to A$ and $\pi_2\colon A \times B \to B$ are both continuous maps. We will discuss π_1; the other case is similar. Let $U \subset A$ be an open set. Then $(\pi_1)^{-1}(U) = U \times B$, and Lemma 1.5 implies that this latter set is open in $A \times B$. Hence π_1 is continuous. ◊

The following lemma gives a simple variant on the definition of continuity, which turns out to be equivalent to it.

Lemma 7.1.3. *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be sets, and let $f\colon A \to B$ be a map. The map f is continuous iff for every closed subset $C \subset B$, the set $f^{-1}(C)$ is closed in A.*

Proof. First assume f is continuous. Let $C \subset B$ be closed. Then $B - C$ is open in B, and so $f^{-1}(B - C)$ is open by hypothesis. Using Exercise 1.3.4 we have

$$f^{-1}(B - C) = f^{-1}(B) - f^{-1}(C) = A - f^{-1}(C).$$

Since $A - f^{-1}(C)$ is open, it follows that $f^{-1}(C)$ is closed. We have thus proved one of the implications in the lemma. The other implication is proved similarly. $\square$