3. Infinite Series

A series is an infinite sum of numbers:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]

The individual numbers are called the terms of the series. In the above series, the first term is 1/2, the second term is 1/4, and so on. The nth term is 1/2^n:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots
\]

We can express an infinite series using summation notation. For example, the above series would be written as follows:

\[
\sum_{n=1}^{\infty} \frac{1}{2^n}
\]

The sum symbol means "sum". The n = 1 on the bottom refers to the fact that the first term is n = 1 (some series start with n = 0 or n = 2), and the \( \infty \) on the top indicates that the series continues indefinitely.

The Sum of a Series

Many series add up to an infinite amount. For example:

\[
\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + \cdots = \infty
\]

\[
\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \cdots = \infty
\]

However, if the terms of a series become smaller and smaller, it is possible for the series to have a finite sum. The basic example is the series

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

which sums to 1.
But what exactly does it mean to find the sum of an infinite series? Given a series

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots, \]

a partial sum is the result of adding together the first few terms. For example, the third partial sum is \(a_1 + a_2 + a_3\), and the hundredth partial sum is \(a_1 + a_2 + \cdots + a_{100}\). The partial sums form a sequence:

\[
\begin{align*}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots
\end{align*}
\]

By definition, the sum of the series is the limit of this sequence:

**DEFINITION (SUM OF A SERIES)**

The sum of the series \(\sum_{n=1}^{\infty} a_n\) is the limit of the partial sums:

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (a_1 + \cdots + a_n)
\]

**EXAMPLE 1** Consider the series

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]

The first few partial sums of this series are listed below:

\[
\begin{align*}
s_1 &= \frac{1}{2} \\
s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}
\end{align*}
\]

As you can see, the partial sums are converging to 1. Therefore, according to our definition of the
EXAMPLE 2  Find the sum of the series:

\[
0.3 + 0.03 + 0.003 + 0.0003 + \cdots
\]

SOLUTION  Here are the first few partial sums:

\[
\begin{align*}
 s_1 &= 0.3 \\
 s_2 &= 0.3 + 0.03 = 0.33 \\
 s_3 &= 0.3 + 0.03 + 0.003 = 0.333 \\
 s_4 &= 0.3 + 0.03 + 0.003 + 0.0003 = 0.3333
\end{align*}
\]

As you can see, the partial sums are converging to the repeating decimal 0.3333…, which is equal to 1/3.

We use the same terminology for sums of series that we do for limits of sequences and for improper integrals:

CONVERGENCE AND DIVERGENCE

We say the series \( \sum a_n \) **converges** if its sum is a real number.

If the sum is infinite or does not exist, then the series \( \sum a_n \) **diverges.**

Keep in mind that a series can only converge if its terms get smaller and smaller. That is, the sum \( \sum_{n=1}^{\infty} a_n \) can only be finite if \( \lim_{n \to \infty} a_n = 0. \)

DO THE TERMS APPROACH ZERO?

1. If \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) **diverges.**

2. If \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) **may converge, or it may diverge.**
EXAMPLE 3  Find the sum of the series:

\[ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots \]

SOLUTION  Since the individual terms of the series are getting closer and closer to 1, the sum of the series is infinite (for the same reason that \( 1 + 1 + 1 + \cdots = \infty \)).

EXAMPLE 4  Determine whether the series \( \sum_{n=1}^{\infty} \frac{n}{2^n + 1} \) converges or diverges.

SOLUTION  Since \( \lim_{n \to \infty} \frac{n}{2^n + 1} = \frac{1}{2} \neq 0 \), this series diverges to \( \infty \).

By the way, it is quite possible for the sum of a series to be infinite even if the terms get smaller and smaller. For example, even though the terms of the series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \]

become smaller and smaller, the sum of this series is infinite! This series is important enough to have its own name: the harmonic series (named for the frequencies of harmonic overtones in music). You should always remember that the harmonic series diverges.

Geometric Series

A geometric series is the sum of the terms of a geometric sequence. For example, the series

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \]

is geometric, with a common ratio of 1/2 (i.e. each term is 1/2 times the previous term). Here are several more examples:

\[ 12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots \quad \text{(common ratio of 1/3)} \]

\[ 1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots \quad \text{(common ratio of 2/5)} \]

\[ 3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots \quad \text{(common ratio or 2/3)} \]

In our study of geometric sequences, we learned that the formula for the \( n \)th term of a geometric sequence has the form \( ar^n \), where \( r \) is the common ratio. If we start the series at \( n = 0 \), then the constant \( a \) can be interpreted as the value of the first term:
GEOMETRIC SERIES FORMULA

The formula for a geometric series is \( \sum_{n=0}^{\infty} ar^n \), where \( a \) is the first term and \( r \) is the common ratio.

EXAMPLE 5   Rewrite the series

\[
3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots
\]

using summation notation.

SOLUTION   The first term of the series is 3, and each term is \( \frac{2}{3} \) of the previous term. Therefore:

\[
3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots = \sum_{n=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^n
\]

Notice that the terms of a geometric series \( \sum_{n=0}^{\infty} ar^n \) only approach 0 if the ratio \( r \) is between \(-1\) and 1. It turns out that any such series converges:

The geometric series \( \sum_{n=0}^{\infty} ar^n \) converges for \(|r| < 1\) and diverges for \(|r| \geq 1\).

There is a special trick to finding the sum of a geometric series, illustrated in the next several examples:

EXAMPLE 6   Find the sum of the series:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]

SOLUTION   We already know that this sum is 1, but this series is useful for illustrating the general method. Let \( x \) be the (unknown) sum of the series:

\[
x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]

If we multiply this equation by \( \frac{1}{2} \) (the common ratio), each term of the sum is halved:

\[
\frac{1}{2}x = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

But the sum on the right is just the later portion of the original series:
This gives us the equation
\[ x = \frac{1}{2} + \frac{1}{2}x \]
and therefore \( x = 1 \). We conclude that the sum of the series is 1.

**EXAMPLE 7**  Find the sum of the series:
\[
1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots
\]

**SOLUTION**  Let \( x \) be the unknown value of the sum:
\[
x = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots
\]
If we ignore the initial 1, the remaining terms of the series are simply \( \frac{2}{3}x \):
\[
x = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots
\]
This gives us the equation
\[ x = 1 + \frac{2}{3}x \]
and therefore \( x = 3 \). We conclude that the sum of the series is 3.

This trick only works for geometric series. The key idea is that a geometric series is *self-similar*, in the sense that the portion of the series after the first term is simply a multiple of the whole series.

By the way, a similar trick can be used to convert any repeating decimal to a fraction:

**EXAMPLE 8**  Express 0.48484848\ldots as a fraction of integers.

**SOLUTION**  Let \( x = 0.48484848\ldots \). Then \( x \) can be written as a sum of two parts:
\[ x = 0.48 + 0.00484848\ldots \]
The first part is the same as \( 48/100 \), and the second part is actually \( x/100 \). This gives us the equation
\[
x = \frac{48}{100} + \frac{x}{100}
\]
Solving for $x$ yields 
\[ x = \frac{48}{100} + \frac{1}{100}x. \]

This trick not only looks the same, it is the same. The decimal in the previous example is by itself a geometric series:

\[ 0.48484848\ldots = \frac{48}{100} + \frac{48}{10,000} + \frac{48}{1,000,000} + \cdots = \sum_{n=1}^{\infty} \frac{48}{(100)^n} \]

Similar reasoning can sometimes be used to evaluate infinite algebraic expressions that aren't series:

**EXAMPLE 9** Find the value of the expression \( \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} \}. \)

**SOLUTION** Observe that this expression contains itself. In particular, if we let

\[ x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} \]

then

\[ x = \sqrt{6 + x}. \]

Squaring and solving for $x$ yields $x = 3$ or $x = -2$. Since the square root must be positive, we conclude that $x = 3$.

You can verify this for yourself on your calculator. Starting with any number, alternately add 6 and take the square root. The more times you repeat these steps, the closer the result will be to 3.

**EXAMPLE 10** Find the value of the expression \( 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \).

**SOLUTION** This kind of expression is known as a continued fraction. If we let $x$ be the value of the entire expression, then

\[ x = 2 + \frac{1}{x} \]

Solving for $x$ yields $x = 1 \pm \sqrt{2}$. Rejecting the negative solution as nonsensical, we conclude that the continued fraction above is equal to $1 + \sqrt{2}$. 
**General Formula for a Geometric Series**

It is possible to avoid repeating this same trick over and over by doing it once in general. Given the series

\[ \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots \]

let \( x \) be the unknown value of the sum:

\[ x = a + ar + ar^2 + ar^3 + \cdots \]

Ignoring the initial \( a \), the rest of the sum is equal to \( rx \):

\[ x = a + \frac{ar + ar^2 + ar^3 + \cdots}{rx} \]

This gives us the equation

\[ x = a + rx. \]

Solving for \( x \) yields the formula

\[ x = \frac{a}{1-r}. \]

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**SUM OF A GEOMETRIC SERIES**

If \( |r| < 1 \), then:

\[ \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \]

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**EXAMPLE 11**

Find the sums of the following series:

(a) \[ \sum_{n=0}^{\infty} \frac{5}{3^n} \]

(b) \[ 8 + 6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \cdots \]

**SOLUTION**

(a) We have \[ \sum_{n=0}^{\infty} \frac{5}{3^n} = \sum_{n=0}^{\infty} 5 \cdot \left(\frac{1}{3}\right)^n = \frac{5}{1-\frac{1}{3}} = \frac{15}{2}. \]

(b) The first term of this series is \( a = 8 \), and the ratio between terms is \( r = \frac{3}{4} \). Therefore, the sum is

\[ \frac{a}{1-r} = \frac{8}{1-\frac{3}{4}} = 32. \]

You should be familiar with both methods for summing a geometric series (using the trick or using the formula).
Combining Series

Because infinite series and integrals are two different sorts of infinite sums, they obey many of the same rules:

**RULES FOR SERIES**

1. \[ \sum_{n=1}^{\infty} \alpha a_n = \alpha \sum_{n=1}^{\infty} a_n \] (where \( \alpha \) is a constant)

2. \[ \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \]

3. \[ \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \]

**EXAMPLE 12** Find the sum of the series \( \sum_{n=0}^{\infty} \left( \frac{1}{2^n} + \frac{5}{3^n} \right) \).

**SOLUTION** We can break this series up into two geometric series:

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2^n} + \frac{5}{3^n} \right) = \sum_{n=0}^{\infty} \frac{1}{2^n} + \sum_{n=0}^{\infty} \frac{5}{3^n}
\]

\[
= \frac{1}{1 - 1/2} + \frac{5}{1 - 1/3} = 2 + \frac{15}{2} = \frac{19}{2}
\]
EXERCISES

1–6  Rewrite the given series using summation notation.

1. \( \frac{1}{5} + \frac{2}{6} + \frac{4}{7} + \frac{8}{8} + \frac{16}{9} + \cdots \)

2. \( 2 + \frac{5}{2} + \frac{10}{6} + \frac{17}{24} + \frac{26}{120} + \cdots \)

3. \( 9 + \frac{9}{2} + \frac{9}{4} + \frac{9}{8} + \frac{9}{16} + \cdots \)

4. \( 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots \)

5. \( 9 + 6 + 4 + \frac{8}{3} + \frac{16}{9} + \cdots \)

6. \( \frac{1}{10} + \frac{1}{5} + \frac{2}{5} + \frac{4}{5} + \frac{8}{5} + \cdots \)

7–14  Find the sum of the given geometric series.

7. \( 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots \)

8. \( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \cdots \)

9. \( \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243} + \cdots \)

10. \( 10 + 5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots \)

11. \( \frac{1}{18} + \frac{1}{6} + \frac{1}{2} + \frac{3}{2} + \frac{9}{2} + \cdots \)

12. \( \frac{1}{24} + \frac{1}{18} + \frac{2}{27} + \frac{8}{81} + \cdots \)

13. \( 9 + 6 + 4 + \frac{8}{3} + \frac{16}{9} + \cdots \)

14. \( 10 + 4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots \)

15. Express the number 0.151515… as a fraction of integers.

16. Express the number 0.135135135… as a fraction of integers.

17–18  Find the value of the given self-similar expression.

17. \( 3 + \frac{1}{3} \left( 3 + \frac{1}{3} \left( 3 + \frac{1}{3} \left( \cdots \right) \right) \right) \)

18. \( 4 + \frac{1}{2 + \frac{1}{4 + \frac{1}{2 + \cdots}}} \)

19–28  Determine whether the given series is convergent or divergent. If it is convergent, find the sum.

19. \( \sum_{n=0}^{\infty} \frac{2^n}{5^n} \)

20. \( \sum_{n=1}^{\infty} \frac{1}{3^n} \)

21. \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2} + 1} \)

22. \( \sum_{n=1}^{\infty} (-1)^n \)

23. \( \sum_{n=0}^{\infty} \frac{1}{e^{2n}} \)

24. \( \sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} \)

25. \( \sum_{n=1}^{\infty} \tan^{-1}(n) \)

26. \( \sum_{n=1}^{\infty} \frac{n^3}{n(n+2)} \)

27. \( \sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} \)

28. \( \sum_{n=1}^{\infty} [2(0.1)^n + (0.2)^n] \)